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Notes on the Penny-Weighing Problem, Etc.

On Sampling the Zeros of Bandwidth Limited Signals

Enhancement of Pulse Train Signals by Comb Filters

Non-Mean-Square Error Criteria

A Comment on Pattern Redundancy

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INFORMATION THEORY

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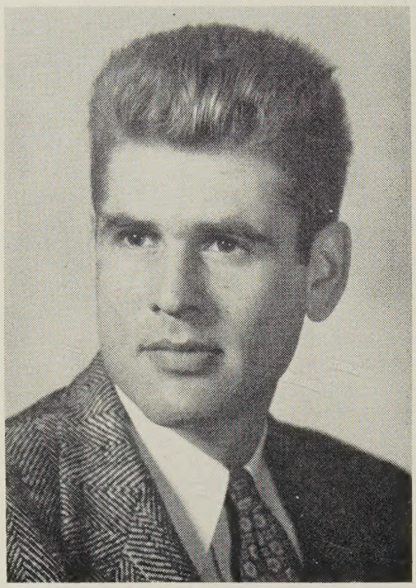

Peter Elias

Peter Elias (S'48—A'51—M'56) was born on November 26, 1923, in New Brunswick, N. J. He received the S. B. degree from the Massachusetts Institute of Technology, Cambridge, in 1944, and the degrees of M. A., M.E.S., and Ph.D. in 1948, 1949, and 1950, respectively, from Harvard University, Cambridge, Mass. He served with the U.S. Navy as a radio technician from 1944 to 1946.

From 1950 to 1953, Dr. Elias was a Junior Fellow in the Society of Fellows at Harvard University doing research on information theory. He became an Assistant Professor at M.I.T. in 1953, in the Electrical Engineering Department and Research Laboratory of Electronics. In 1956, he became an Associate Professor. He has been working on two-dimensional random processes and pictures as information sources, efficient coding of redundant sources, and noisy channel coding problems.

Dr. Elias is a member of Sigma Xi, Eta Kappa Nu and the Institute of Mathematical Statistics. He was chairman of the 1956 M.I.T.-PGIT Information Theory Symposium, and currently is chairman of the IRE Technical Committee on Information Theory and Modulation Systems. He was recently elected a member of the PGIT Administrative Committee.

Two Famous Papers

PETER ELIAS

It is common in editorials to discuss matters of general policy and not specific research. But the two papers I would like to describe have been written so often, by so many different authors under so many different titles, that they have earned editorial consideration.

The first paper has the generic title "Information Theory, Photosynthesis and Religion" (title courtesy of D. A. Huffman), and is written by an engineer or physicist. It discusses the surprisingly close relationship between the vocabulary and conceptual framework of information theory and that of psychology (or genetics, or linguistics, or psychiatry, or business organization). It is pointed out that the concepts of structure, pattern, entropy, noise, transmitter, receiver, and code are (when properly interpreted) central to both. Having placed the discipline of psychology for the first time on a sound scientific base, the author modestly leaves the filling in of the outline to the psychologists. He has, of course, read up on the field in preparation for writing the paper, and has a firm grasp of the essentials, but he has been anxious not to clutter his mind with such details as the state of knowledge in the field, what the central problems are, how they are being attacked, et cetera, et cetera, et cetera.

There is a constructive alternative for the author of this paper. If he is willing to give up larceny for a life of honest toil, he can find a competent psychologist and spend several years at intensive mutual education, leading to productive joint research. But this has some disadvantages from his point of view. First, psychology would not be placed on a sound scientific base for several extra years. Second, he might find himself, as so many have, diverted from the broader questions, wasting his time on problems whose only merit is that they are vitally important, unsolved, and in need of interdisciplinary effort. In fact, he might spend so much time solving such problems that psychology never *would* be placed on a sound scientific base.

The second paper is typically called "The Optimum Linear Mean Square Filter for Separating Sinusoidally Modulated Triangular Signals from Randomly Sampled Stationary Gaussian Noise, with Applications to a Problem in Radar." The details vary from version to version, but the initial physical problem has as its major interest its obvious nonlinearity. An effective discussion of this problem would require some really new thinking of a difficult sort, so the author quickly substitutes an unrelated linear problem which is more amenable to analysis. He treats this irrelevant linear problem in a very general way, and by a triumph of analytical technique is able to present its solution, not quite in closed form, but as the solution to an integral equation whose kernal is the solution to another, bivariate integral equation. He notes that the problem is now in a form in which standard numerical analysis techniques, and one of the micromicrosecond computers which people are now beginning to discuss, can provide detailed answers to specific questions. Many authors might rest here (in fact many do), but ours wants real insight into the character of the results. By carefully taking limits and investigating asymptotic behavior he succeeds in showing that in a few very special cases (which include all those which have any conceivable application or offer any significant insight) the results of this analysis agree with the results of the Wiener-Lee-Zadeh-Raggazzini theory—the very results, indeed, which Wiener, Lee, Zadeh, and Raggazzini obtained years before.

These two papers have been written—and even published—often enough by now.

I suggest that we stop writing them, and release a large supply of manpower to work on the exciting and important problems which need investigation.

An Experimental Investigation of Some Properties of Band-Pass Limited Gaussian Noise*

KJELL BLÖTEKJÆR†

Summary—The probability distribution of time intervals between successive zero crossings of band-pass limited Gaussian noise is determined experimentally for a number of different filters having nearly rectangular frequency characteristics. For one particular filter the distribution of time intervals between crossings of levels different from zero is also found.

Introduction

The DISTRIBUTION of time intervals between successive crossings of any level, and more particularly the zero level, of band-pass limited Gaussian noise cannot be derived explicitly by analytical methods, and numerical evaluation tends to become very involved.

The purpose of the present investigation is to determine this distribution by experiment, to study its relation to the noise power spectrum, and to discuss the results in view of the solution of a related problem which can be solved analytically.

Rice[1] presents a probability density function of time intervals between successive zero crossings obtained experimentally by M. E. Campbell from a limited amount of data. The present work is based on a different measuring technique, and supplements and extends Campbell's results.

Principles of Measuring Technique

The noise signal analyzed was derived from an ordinary video amplifier with a flat noise power spectrum; the desired noise spectrum was obtained by combining this noise generator with a low-pass and a high-pass filter having variable frequency limits.

The analysis was carried out directly on the noise signal. A Schmitt trigger was used to convert the noise into a square wave, see Fig. 1(b). The zero crossings of this square wave coincide with the zeros of the noise waveform. These square waves were again converted into a sawtooth wave, shown in Fig. 1(c). The slopes of the sawtooth pulses were maintained constant, the height of these pulses then being proportional to the time interval between successive crossings of the zero level. By applying the sawtooth waveform to another Schmitt trigger and an electronic counter, it is possible to count the number of time intervals exceeding a certain preset value. In this manner the integrated probability distribution was derived, and hence, by straightforward manipulations, the desired probability density function is obtained.

Discussion of Errors Involved in the Measuring Technique

Two main sources of error are involved: statistical fluctuations, and systematic errors due to imperfections in the measuring equipment.

Purely statistical fluctuations can be made as small as desired by increasing the number of observations N, *i.e.*, the number of counts, because the relative fluctuations are inversely proportional to the square root of N.

The remaining limitation on the accuracy of the measurements is due to imperfections in the experimental technique described above. A thorough analysis of the measuring technique revealed that the relative uncertainty was between 1 and 2 per cent. It was necessary therefore to choose the number of counts sufficiently large to make the statistical fluctuations smaller than this percentage.

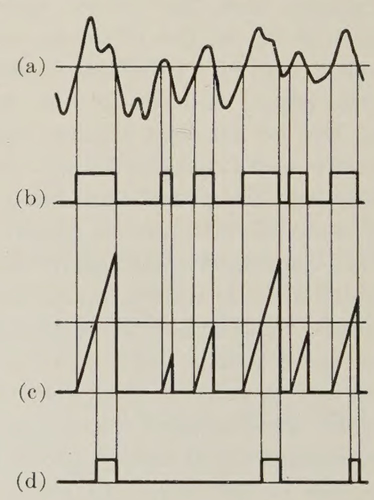

Fig. 1—Principle of measuring technique. (a) Original noise signal. (b) Square wave derived from (a). (c) Sawtooth pulses derived from (b). (d) Square pulses indicating that the sawtooth pulses (c) exceed a preselected level.

Results

The distribution of time intervals between successive zero crossings was determined for a number of different filters. Each curve was found from approximately 30 measured points.

* Manuscript received by the PGIT, September 30, 1957.

† Norwegian Defense Research Establishment, Kjeller, Norway. This work was carried out as partial fulfillment of the final examination at the Norwegian Institute of Technology.

[1] S. O. Rice, "Mathematical analysis of random noise," *Bell Sys. Tech. J.*, vol. 23, p. 282; July 1944, and vol. 24, p. 46; January 1945.

In order to compare the experimental results with theory, it is desirable to use filters having frequency characteristics which can be expressed in a simple mathematical form. In the present investigation it was aimed at obtaining rectangular frequency characteristics. The actual slopes of the edges were approximately 200 db per octave, while the intended flat portion of the filter characteristics had irregular variations of maximally 2.5 db. Thus, exact mathematical expressions for the noise power spectrum could not be obtained, and the assumption of rectangular spectra is a relatively crude approximation. This must be borne in mind when theoretical and experimental results are compared.

At low frequencies the power spectrum of the noise source is no longer flat due to various reasons (flicker effect, etc.). Accordingly, it was found necessary to eliminate this part of the spectrum by filtering. Therefore, noise with a power spectrum starting at zero frequency could not be examined, but to obtain a reasonable approximation the lower frequency limit of the filter was chosen as low as permissible, at 440 cps. The higher frequency limit was 10, 250 cps, which was well below the maximum frequency that could be analyzed by the measuring equipment.

periodic component of frequency f_0 would be a δ function at $\varphi = \pi$. When the lower frequencies are added, the distribution becomes broader, and the maximum moves towards longer time intervals. The second maximum at $\varphi \approx 9$ is due to the case where the noise signal has two maxima and one minimum between successive zero crossings.

The dotted curve in Fig. 2 represents a probability density function which was calculated by Rice. It gives the probability density of a noise signal $I(t)$ crossing the zero level with a negative slope at time $t + \tau$, given that $I(t)$ is crossing the zero level at the time t with positive slope.

Apart from time intervals between sucessive zero crossings, this distribution also includes time intervals with intermittent zero crossings. Therefore, for all values of φ this distribution should be situated above the experimental curve. Rice has shown, using theoretical arguments, that the two curves should practically coincide for values of φ less than three. Fig. 2 shows that this is in fact the case.

For somewhat higher values of φ, however, the experimentally obtained distribution exceeds the theoretical distribution found by Rice. This might be due partly to the low-frequency cutoff in the noise spectrum, and partly to

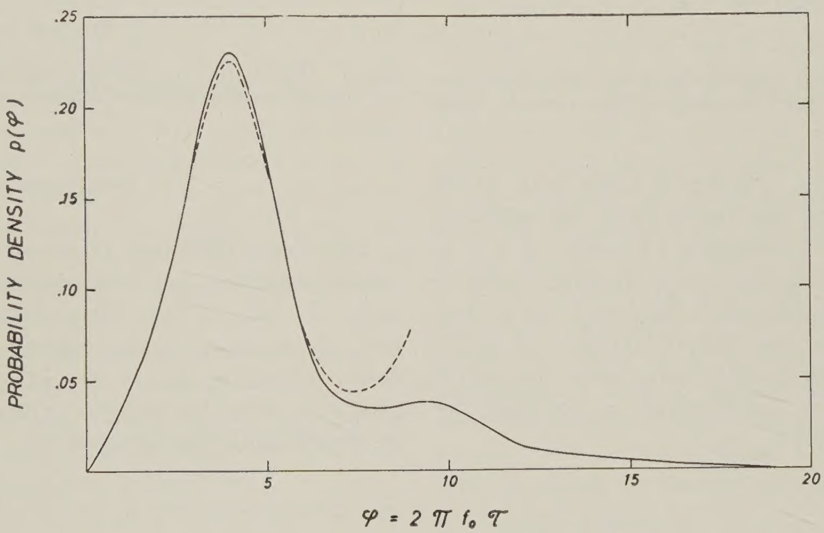

Fig. 2—Distribution of time intervals between successive zero crossings. Filter: approximately low pass. Related theoretical curve shown by dotted line.

The distribution of time intervals between successive zero crossings for noise passed through this filter is shown in Fig. 2, where the abscissa is expressed in terms of $\varphi = 2\pi f_0 \tau$, where f_0 is the upper frequency limit of the filter. In transforming from τ to φ all distributions $p(\varphi)$ become equal, no matter the value of f_0, provided the shape of the power spectrum is the same. This distribution has a distinct maximum at $\varphi \approx 4$ and a less distinct maximum at $\varphi \approx 9$. The physical reasons for these maxima are as follows. The corresponding distribution for a

the irregular variations over the pass band of the filter characteristic.

Fig. 3 gives the results of measurements with several filters with upper frequency limits at 10,250 cps, and with variable lower frequency limits. In this figure it should be noted that the vertical scale is enlarged for values of φ larger than 6.5.

As the lower frequency limit is increased, rendering the noise spectrum narrower, the distributions become steeper round a single maximum which is moving towards

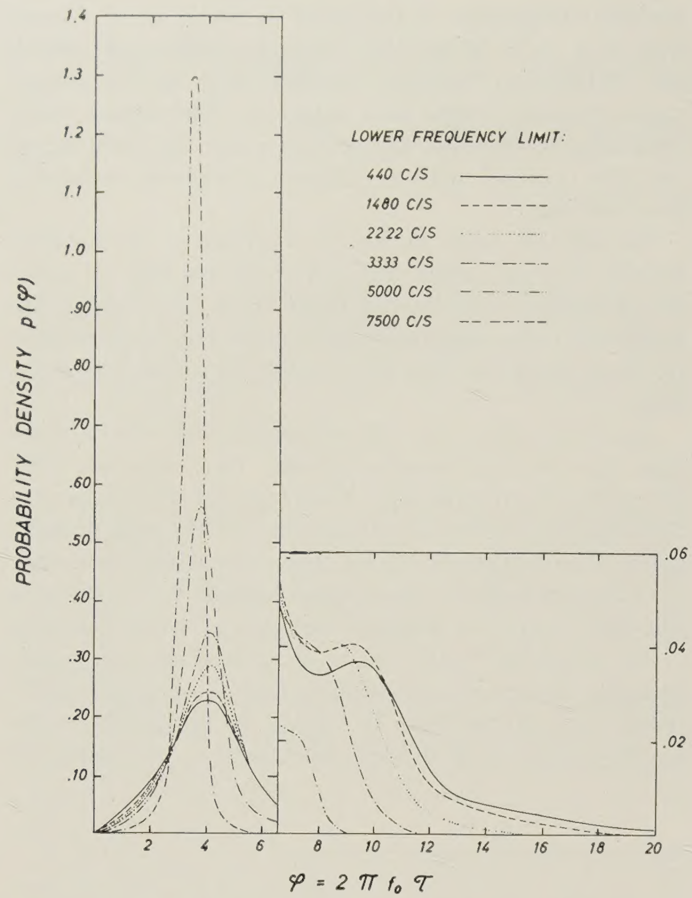

Fig. 3—Distribution of time intervals between successive zero crossings. Filters: fixed upper frequency limit, variable lower frequency limit.

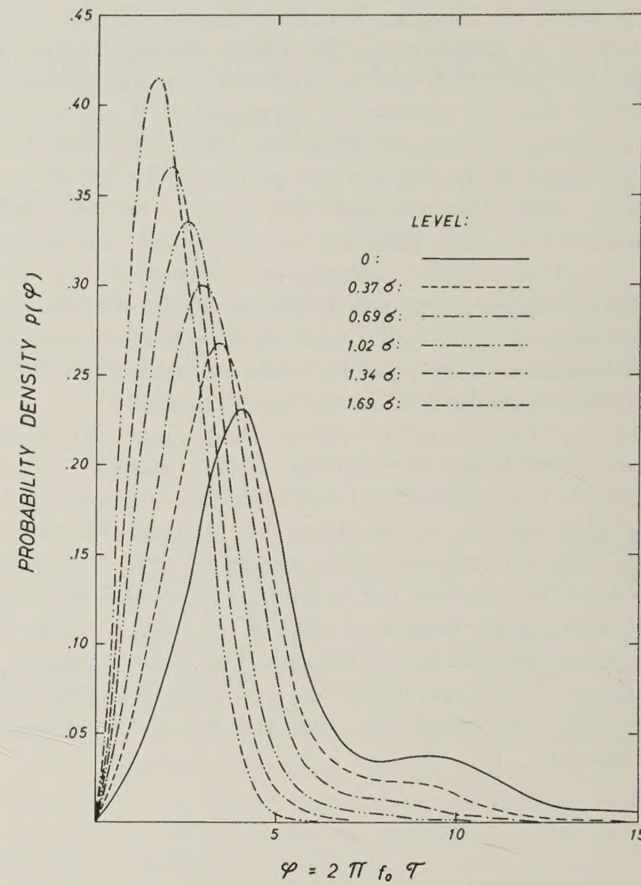

Fig. 4—Distribution of time intervals between successive crossings of some different levels. Filter: approximately low pass.

shorter time intervals (*i.e.*, smaller φ). This is in agreement with the fact that the distribution for infinitely narrow filters should tend towards a δ function at $\varphi = \pi$.

Fig. 4 gives the distributions of time intervals between successive crossings of levels other than zero, for a filter with frequency limits 440 cps and 10,250 cps. All distributions are normalized to give the same area. The levels examined were all positive, and between the crossings the signal was above the level.

Conclusion

The main difficulty in comparing the experimental results with the related theoretical results of Rice appears to be the lack of ideal filter characteristics, and not the lack of accuracy in the experimental method. Further work to test the theory should therefore concentrate on producing filter characteristics which can be represented by simple analytical expressions.

Notes on the Penny-Weighing Problem, Lossless Symbol Coding with Nonprimes, Etc.*

MARCEL J. E. GOLAY†

Summary—The method of construction of lossless symbol coding matrices for one-error correction is illustrated for the case when the prime symbol order is three, and the application of this matrix to the penny-weighing problem is described. This method is then extended to those cases in which the symbol order is 2^2, 2^3, 2^4, 2^6, 3^2, 3^3, 3^4, 3^5, 5^2, 5^3, 5^4, 7^2, 7^3, and p^2, where p is any higher prime. This extension is based on the concept of the master iterating matrix. These matrices are given for the first thirteen cases cited, and their existence is demonstrated for p^2.

This paper concludes with a short description of Zaremba's condition, and its application to various problems, and more particularly to the hypothetical one-error correcting close-packed code with the symbol order 6.

Introduction

In a former note,[1] the writer described the construction of lossless symbol coding[2] matrices for one-error correction when the symbol order is prime, and two additional singular cases of close-packed coding for two and three-error correction, respectively.

An elaboration of the one-error correcting binary code was published in an industrial journal,[3] but the only additional error correcting lossless codes (with no restriction placed on the individual symbol errors) published since 1949 are due to Zaremba,[4] who showed on group-theoretical grounds that a close-packed code book for message coding[2] always exists for one-error correction, when the symbol order is a power of a prime. The essential portion of these notes will be devoted to those cases of lossless nonprime symbol coding for which coding matrices can be constructed systematically.

In the first part of the discussion, a recapitulation is given of the matrix construction formerly described,[1] for lossless one-error correction coding when the symbol order is three, and the application of this particular code to the penny-weighing problem is described.

In the second part, it is shown how the construction of lossless symbol coding matrices may be extended from code symbols of prime orders to code symbols of orders 2^2, 2^3, 2^4, 2^6, 3^2, 3^3, 3^4, 3^5, 5^2, 5^3, 5^4, 7^2, 7^3, and any square of an odd prime number.

The third part of the discussion contains observations on various solved or unsolved coding problems.

The Penny-Weighing Problem

Given a balance which can be used to determine whether two weights are equal, or which is heavier, and given a certain quantity of pennies, of which at most one may be heavier or lighter than the standard weight, it is asked to determine what paired assemblies of pennies should have their weight compared with each other, in order to find, with a minimum number of operations, which penny, if any, is too heavy or too light. It is also required that the weighing program be completely predetermined, and thus be not affected by the results of the successive weighings.

Each penny may be in one of three possible states, too light, of correct weight, and too heavy, and its state is thus expressible by a ternary symbol. The information gathered from each weighing is also expressible by a ternary symbol. These circumstances suggest an analogy with the transmission and reception of messages composed of ternary symbols, of which one at most may be received in error. It may be surmised that, for instance, three weighings, yielding $\log_2 27$ information bits, should determine which of 13 pennies, if any, is too light or too heavy. A form of lossless coding appears required to solve this problem, since each penny may be off-standard in two ways, thus yielding 26 possibilities, to which must be added the twenty-seventh possibility of all being of standard weight.

Consider now the coding matrices described in the former publication,[1] for the case $p = 3$. The simplest matrix, corresponding to $n = 1$, covers the trivial case of no information message and a single transmitted ternary check symbol, X_1. The coding matrix for this case consists of a single "1":

$$\begin{array}{|c} X_1 \\ \hline 1 \end{array} \qquad (1)$$

and since there are no information symbols, the transmitted check symbol will be zero.

The passage from $n = 1$ to $n = 2$ is made by writing the single term matrix (1) three times, writing below the numbers 2, 1 and 0, and adding a $\begin{smallmatrix}0\\1\end{smallmatrix}$ column at the end:

* Manuscript received by the PGIT, February 25, 1938; revised manuscript received June 15, 1958.

† Philco Corporation, Philadelphia, Pa.

[1] M. J. E. Golay, "Notes on digital coding," Proc. IRE, vol. 37, p. 657; June, 1949.

[2] The expressions "symbol coding," "message coding," "corrector," "characteristic," etc., are given the same meaning as in a former publication (M. J. E. Golay, "Binary coding," IRE Trans. on Information Theory, no. PGIT-4, pp. 23–28; September, 1954). A "lossless symbol coding matrix" yields a "close-packed code," but cases are conceivable—although unknown—in which a close-packed code exists but symbol coding is impossible.

[3] R. W. Hamming, "Error detecting and correcting codes," *Bell Sys. Tech. J.*, vol. 29, pp. 147–161; April, 1950. Mention should be made of an earlier article by Shannon, in which the "somewhat artificial" case of coding seven-bit words against one error by means of three parity checks is described and attributed to Hamming (C. E. Shannon, "A mathematical theory of communication," *Bell Sys. Tech. J.*, vol. 27, p. 418; July, 1948.)

[4] S. K. Zaremba, "Covering problems concerning abelian groups," *J. Lond. Math. Soc.*, vol. 27, pp. 242–246; April, 1952.

$$
\begin{array}{cc|cc}
Y_1 & Y_2 & X_1 & X_2 \\
\hline
1 & 1 & 1 & 0 \\
2 & 1 & 0 & 1
\end{array} \qquad (2)
$$

The passage from $n = 2$ to $n = 3$ is accomplished similarly:

$$
\begin{array}{cccccccccc|ccc}
Y_1 & Y_2 & Y_3 & Y_4 & Y_5 & Y_6 & Y_7 & Y_8 & Y_9 & Y_{10} & X_1 & X_2 & X_3 \\
\hline
1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\
2 & 1 & 0 & 1 & 2 & 1 & 0 & 1 & 2 & 1 & 0 & 1 & 0 \\
2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1
\end{array} \qquad (3)
$$

and so forth.

The coding equation:[1]

$$
E_m \equiv X_m + \sum^{k} a_{mk} Y_k \equiv 0 \pmod{p} \qquad (4)
$$
$$
k = 1, \cdots \frac{p^n - 1}{p - 1} - n, \quad m = 1, \cdots n
$$

is used to calculate the X check symbols at the transmitting end, and to calculate the corrector E_m at the receiving end; the terms of matrix (3) are the coefficients of the Y's and X's in (4), for the case $n = 3$, $p = 3$.

The 13 ternary numbers written vertically in the 13 columns of (3), which are termed the "base characteristics" of the X's or Y's, and the other 13 numbers obtained by multiplying the individual digits of the base characteristics by 2 modulo 3, which are termed the "derived characteristics" of the X's or Y's, are all different and represent the two ways in which one of the 13 X's or Y's could have been received in error. Inspection of the corrector, when it is not $\begin{smallmatrix}0\\0\\0\end{smallmatrix}$, indicates which symbol was received in error, and whether the error was $+1$ (base characteristic) or $+2$ (derived characteristic).

The one-to-one correspondence between any nonzero corrector, and the base or derived characteristic of one of the 13 message symbols constitutes the lossless property of this one-error correction code, and it will be readily seen that the iteration process described above conserves this property when passing from n to $n + 1$.

Consider now that the X's and Y's represent 13 pennies, all equal when shipped, but one of which may have had its weight altered upward or downward before being received. This corresponds to the case of a known transmitted message which is not delivered as such to the addressee. Instead, the addressee is given the E_m's obtained from the three weighing operations, and asked which penny, if any, had its weight increased or decreased.

Inspection of the coding matrix indicates a difficulty in determining the E_m's by weighing various assemblies of pennies: none of the matrix lines contains 1's and 2's in equal number. On the other hand, if an extra penny of correct weight is available—a "catalyzer" conveying no information as such—it can be placed on one side of the scale together with Y_1, Y_2, Y_3 and Y_4, while Y_5, Y_6, Y_7, Y_8 and X_3 are placed on the other. The result of this weighing operation clearly yields the addition modulo 3 required to determine E_3.

The weighing operations required to determine E_2 and E_1 would require three extra and nine extra (good) pennies, respectively, but we may obtain instead the sums E_2E_3 and E_1E_3. When the elements of E_2 and E_3 are added modulo 3, the following 13 numbers are obtained under $Y_1 \cdots X_3$: 1020021221011. Likewise, the E_1E_3 numbers are: 0002222111101. This indicates that the determination of E_2E_3 and E_1E_3 can be effected by weighing operations in which, as for E_3, five received pennies are weighed against four other received pennies and the extra (good) penny. These three weighing operations determine completely which penny, if any, is too light or too heavy.

Lossless Symbol Coding with Powers of Primes

When the order of the message symbols is a power of a prime, p^q, n check symbols should cover $p^{qn} - 1$ possibilities of transmission errors.

Any received symbol may be in error in $p^q - 1$ ways, and if at most one symbol per message may be received in error, a first condition for lossless coding is that the total number of information and check symbols in a message be

$$
\frac{p^{qn} - 1}{p^q - 1}.
$$

The coding equation for such a case has the general form:

$$
E_{mi} \equiv X_{mi} + \sum^{k,j} a_{mi}^{kj} Y_{kj} \equiv 0 \pmod{p} \qquad (5)
$$
$$
k = 1, \cdots \frac{p^{qn} - 1}{p^q - 1} - n, \; m = 1, \cdots n, \; i, j = 1, \cdots q
$$

where $X_{m1} \cdots X_{mq}$ and $Y_{n1} \cdots Y_{nq}$ are the q elements of the X_m symbol and Y_n symbol, respectively:

$$
\begin{aligned}
X_m &= (X_{m1}, X_{m2}, \cdots X_{mq}) \\
Y_k &= (Y_{k1}, Y_{k2}, \cdots Y_{kq}) \qquad (5a) \\
E_m &= (E_{m1}, E_{m2}, \cdots E_{mq}).
\end{aligned}
$$

The matrix formed by the coefficients of the Y's and X's in (5) will be termed the (n, p, q) matrix. When $p = q = 2$, the trivial case of a message containing no information and a single check symbol is represented by the (1, 2, 2) matrix:

$$
\begin{array}{|cc}
X_{11} & X_{12} \\
\hline
1 & 0 \\
0 & 1
\end{array} \qquad (6)
$$

The passage from the (1, 2, 2) matrix to the (2, 2, 2) matrix is analogous to that described earlier and in the former publication[1] for the cases $q = 1$. The matrix (6) is first written four times horizontally and the four "iteration matrices"

$$\begin{matrix}1 & 1\\ 1 & 0,\end{matrix} \quad \begin{matrix}1 & 0\\ 0 & 1,\end{matrix} \quad \begin{matrix}0 & 1\\ 1 & 1\end{matrix} \quad \text{and} \quad \begin{matrix}0 & 0\\ 0 & 0\end{matrix} \tag{7}$$

are written below the four identical matrices first written. The (2, 2, 2) matrix is completed by a double column consisting of an all-zero matrix above a (1, 2, 2) matrix.

$$\begin{array}{cccccc|cccc}
Y_{11} & Y_{12} & Y_{21} & Y_{22} & Y_{31} & Y_{32} & X_{11} & X_{12} & X_{21} & X_{22}\\
\hline
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0\\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0\\
1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0\\
1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1.
\end{array} \tag{8}$$

It can be verified by inspection of matrix (8) that all 15 possible nonzero correctors are reproduced by the 10 columns of (8), which are the 10 base characteristics of the 5 pairs of Y's and X's, and by the 5 derived characteristics formed by the sums of the two base characteristics of a $Y_{k1}Y_{k2}$ pair or of an $X_{m1}X_{m2}$ pair. Thus, all 15 possible cases of an error in one of 5 transmitted biquadratic symbols are covered exactly once, and the code calculable with (8) is close packed.

When passing from biquadratic to quaternary symbols, the following convention will be used:

$$00 \rightarrow 0 \quad 01 \rightarrow 1 \quad 10 \rightarrow 2 \quad 11 \rightarrow 3 \tag{9}$$

With this convention, the 64 messages belonging to the (2, 2, 2) code, and determined by giving the Y symbols all combinations of numerical values and calculating the X's with matrix (8), can be written compactly:

00000 02022 10012 12030 20023 22001 30031 32013

00113 02131 10101 12123 20130 22112 30122 32100

00221 02203 10233 12211 20202 22220 30210 32232

00332 02310 10320 12302 20311 22333 30303 32321 (10)

01011 03033 11003 13021 21032 23010 31020 33002

01102 03120 11110 13132 21121 23103 31133 33111

01230 03212 11222 13200 21213 23231 31201 33223

01323 03301 11331 13313 21300 23322 31312 33330.

When passing from the (2, 2, 2) matrix to the (3, 2, 2) matrix, the (2, 2, 2) matrix is written four times horizontally and a third double line is added, consisting of the four iteration matrices, repeated each five times. A twenty-first double column, consisting of two zero matrices above a (1, 2, 2) matrix completes the (3, 2, 2) matrix.

It has been observed that the 15 nonzero correctors associated with the (2, 2, 2) matrix are reproduced by the single columns of the (2, 2, 2) matrix or by the sums of column pairs. A set of three further observations will now be made, namely: that the first column of the four iteration matrices are all different from each other $\left(\begin{matrix}1\\1\end{matrix}, \begin{matrix}1\\0\end{matrix}, \begin{matrix}0\\1\end{matrix} \text{ and } \begin{matrix}0\\0\end{matrix}\right)$; that the second column of all iteration matrices are all different from each other, $\left(\begin{matrix}1\\0\end{matrix}, \begin{matrix}0\\1\end{matrix}, \begin{matrix}1\\1\end{matrix} \text{ and } \begin{matrix}0\\0\end{matrix}\right)$, and that the sums of the column pairs of the four iteration matrices are all different from each other $\left(\begin{matrix}0\\1\end{matrix}, \begin{matrix}1\\1\end{matrix}, \begin{matrix}1\\0\end{matrix} \text{ and } \begin{matrix}0\\0\end{matrix}\right)$. Thus any base or derived characteristic of the (2, 2, 2) matrix is continued, in the (3, 2, 2) matrix, by one of the four biquadratic elements $\begin{matrix}1\\1\end{matrix}, \begin{matrix}1\\0\end{matrix}, \begin{matrix}0\\1\end{matrix}$ or $\begin{matrix}0\\0\end{matrix}$. Therefore, it is concluded from these four observations that the 20 first pairs of X and Y columns of the (3, 2, 2) matrix, and the 20 sums of the two columns of a pair, constitute the 60 different base and derived characteristics of the (3, 2, 2) matrix in which the first four binary symbols do not all vanish. The three additional characteristics of the (3, 2, 2) matrix in which the first four binary symbols vanish are the two columns of the added twenty-first pair, and their sum. Thus, we obtain in all the $p^{qn} - 1 = 2^{2 \cdot 3} - 1 = 63$ characteristics of the (3, 2, 2) matrix.

It is seen readily that the iteration process just described is valid for the passage from any $(n, 2, 2)$ matrix thus formed to another lossless $(n + 1, 2, 2)$ symbol coding matrix for one-error correction. It is also seen that the iteration matrices, which play a part analogous to that played by the single symbols of the last line when passing from an $(n, p, 1)$ matrix to an $(n + 1, p, 1)$ matrix, constitute the key to the iteration process. These iteration matrices, and some associated concepts, are now examined in some detail.

Iteration Matrices

Designate by b_1, b_2 and b_1b_2 the two base elements, $\begin{matrix}1\\0\end{matrix}$ and $\begin{matrix}0\\1\end{matrix}$, of the iteration matrices, and their sum, $\begin{matrix}1\\1\end{matrix}$. With this convention, the three nonvanishing iteration matrices (7) may be written:

$$b_1b_2 \;\; b_1, \qquad b_1 \;\; b_2 \quad \text{and} \quad b_2 \;\; b_1b_2.$$

Consider now the matrix:

$$\begin{matrix}b_1 & b_2\\ b_2 & b_1b_2\end{matrix}. \tag{11}$$

It will be observed that the second and third iteration matrices utilized above are the two lines of matrix (11), and that the first iteration matrix is the sum modulo 2 of these two lines.

TABLE I

Master Iteration Matrices

	$q = 2$	$q = 3$		$q = 4$		$q = 5$		$q = 6$	
MAT. IND.	1 2 12	1 2 3 12 23	1 2 3 13 123	1 2 3 4 12 23 34	1 2 3 4 14 124 1234	1 2 3 4 5 12 23 34 45	1 2 3 4 5 15 125 1235 12345	1 2 3 4 5 6 12 23 34 45 56	1 2 3 4 5 6 16 126 1236 12346 123456
$p = 2$	MIM	MIM	MIM	MIM	MIM	α's: 1 0 1 1 0 0 1 0 1 1 β's: 1 1 1 0 0 1 0 0 1 0 1 0 1 0 1	α's: 1 1 0 1 0 0 1 1 0 1 β's: 1 1 1 0 0 1 0 0 1 0 1 0 1 0 1	MIM	MIM
$p = 3$	MIM	MIM	α's: 1 2 2 β's: 1 1 0 1 0 2	MIM	MIM	MIM	α's: 1 2 1 2 2 β's: 1 1 0 0 0 1 0 2 0 0 1 0 0 1 0 1 0 0 0 2 also α's: 1 0 2 2 1 1 1 0 2 0 β's: 1 0 0 1 0 0 1 0 0 1 1 0 1 0 1		
$p = 5$	α's: 1 3 β's: 1 3	α's: 1 4 2 β's: 1 2 0 1 0 1	MIM	MIM	MIM				
$p = 7$	MIM	α's: 1 4 5 β's: 1 0 5 1 3 3	MIM						

Designate by p_{ij} the element at the intersection of the ith line and jth column of matrix (11), and let α_i and β_j designate factors which for $p = 2$ may be 1 or 0. It will be verified by inspection that the set of three observations made earlier is equivalent to the observation that no two sums of the form

$$\sum^{i,j} \alpha_i \beta_j p_{ij} \tag{12}$$

are identical unless all the α's and β's of the two sums are also identical. This last observation is also equivalent to the observation that no sum of the form (12) vanish, unless all α's and β's vanish. It will be convenient to state that matrix (11) fulfils for $p = q = 2$ the general condition:

$$\sum^{i,j} \alpha_i \beta_j p_{ij} \not\equiv 0 \pmod{p}$$

when

$$\sum^{i} \alpha_i \neq 0, \qquad \sum^{j} \beta_j \neq 0 \tag{13}$$

$$i,j = 1, 2, \cdots q \quad 0 \leq \alpha_i < p, 0 \leq \beta_j < p$$

and any matrix satisfying this condition will be termed a "master iteration matrix."[5] A set of p^q iteration matrices is formed from a master iteration matrix by giving the α_i's in $\sum \alpha_i p_{ij}$ all p^q combinations of values. When passing from the (n, p, q) coding matrix to the $(n + 1, p, q)$ coding matrix, the (n, p, q) matrix is written horizontally p^q times, and each iteration matrix is written $(p^{qn} - 1)/(p^q - 1)$ times below each (n, p, q) matrix. The $(n + 1, p, q)$ matrix is completed with a q-tuple column consisting of n all-zero matrices above a $(1, p, q)$ matrix (*i.e.*, a $q \times q$ matrix of zeros except for a main diagonal of 1's).

Table I has been prepared to list, in compact form, several matrices which, depending upon the value of p, may or may not constitute master iteration matrices (MIM).

All matrices listed in Table I are symmetric with respect to their main diagonal, and each sucessive line of these matrices is shifted to the left one element with re-

[5] A master iteration matrix is three-dimensional, since it is a two-dimensional array of one-dimensional sequences. Two-dimensional matrices may be derived from it with the operation $\sum^i \alpha_i p_{ij}$ and one-dimensional sequences may be derived from it with the operation $\sum^{i,j} \alpha_i \beta_j p_{ij}$.

spect to the line above it. They are therefore fully characterized by their first line and last column. Furthermore, the indices only of the q-dimensioned base elements p_r forming the p_{ij}'s of the first line and last column have been indicated in the matrix indices (MAT. IND.) lines. For instance, the first of the two matrices at the $p = 3$, $q = 3$ intersection is the ternary master iteration matrix:

$$\begin{array}{ccc} t_1 & t_2 & t_3 \\ t_2 & t_3 & t_1t_2 \\ t_3 & t_1t_2 & t_2t_3 \end{array} \tag{14}$$

where

$$t_1 = \begin{matrix}1\\0\\0\end{matrix}, \quad t_2 = \begin{matrix}0\\1\\0\end{matrix}, \quad t_3 = \begin{matrix}0\\0\\1\end{matrix}, \quad t_1t_2 = \begin{matrix}1\\1\\0\end{matrix}, \quad t_2t_3 = \begin{matrix}0\\1\\1\end{matrix}.$$

Giving the α_i's in $\sum \alpha_i p_{ij}$ the successive 27 combinations of values:

$$\begin{matrix}2\\2,\\2\end{matrix}\ \begin{matrix}2\\2,\\1\end{matrix}\ \begin{matrix}2\\2,\\0\end{matrix}\ \cdots\ \begin{matrix}0\\0,\\2\end{matrix}\ \begin{matrix}0\\0,\\1\end{matrix}\ \begin{matrix}0\\0\\0\end{matrix}$$

yields the 27 successive iteration matrices:

$$\begin{matrix}222\\211,\\221\end{matrix}\ \begin{matrix}212\\200,\\120\end{matrix}\ \begin{matrix}202\\222,\\022\end{matrix}\ \cdots\ \begin{matrix}020\\022,\\202\end{matrix}\ \begin{matrix}010\\011,\\101\end{matrix}\ \begin{matrix}000\\000.\\000\end{matrix}$$

When a matrix is not a master iteration matrix, the single set or the several sets of values of the α_i's and β_j's for which (13) is not fulfilled are listed in ascending index order. For instance, the second matrix at the $p = 3$, $q = 3$ intersection is not a master iteration matrix because (13) is not satisfied when

$$\alpha_1 = 1, \quad \alpha_2 = 2, \quad \alpha_3 = 2$$

and when

$$\beta_1 = 1, \quad \beta_2 = 1, \quad \beta_3 = 0.$$

Alternately, the same α values and

$$\beta_1 = 1, \quad \beta_2 = 0, \quad \beta_3 = 2$$

do not satisfy (13), nor again the same α values (or the α values multiplied by 2 modulo 3) and any linear combination modulo 3 of the β values given above, such as

$$\beta_1 = 2, \quad \beta_2 = 1, \quad \beta_3 = 2$$

or

$$\beta_1 = 0, \quad \beta_2 = 1, \quad \beta_3 = 1, \quad etc.$$

The remarks made above apply to all sets of values of the α's and β's given in Table I which do not satisfy (13), and since the matrices of Table I are symmetric with respect to their main diagonal, the α's and β's may be interchanged in all sets of values given.

The case $p = 2$, $q = 5$ is particularly noteworthy, since it is the only case examined for which a master iteration matrix was not found. Six other symmetric matrices have been examined, and none has yielded a master iteration matrix for this case. The indices of the first and last line of the matrices thus examined are:

$$\begin{matrix}1 & 2 & 3 & 4 & 5,\\ 5 & 13 & 24 & 35 & 14\end{matrix}\quad \begin{matrix}1 & 2 & 3 & 4 & 5,\\ 5 & 12 & 34 & 15 & 23\end{matrix}\quad \begin{matrix}1 & 2 & 3 & 4 & 5,\\ 5 & 123 & 234 & 345 & 145\end{matrix}$$

$$\begin{matrix}1 & 2 & 3 & 4 & 5,\\ 5 & 12 & 123 & 1234 & 12345\end{matrix}\quad \begin{matrix}1 & 2 & 3 & 4 & 5,\\ 5 & 12345 & 2345 & 345 & 45\end{matrix}\quad \begin{matrix}1 & 2 & 3 & 4 & 5\\ 5 & 125 & 123 & 234 & 345.\end{matrix}$$

It has been noted[2] that no case of lossless coding is known, for which symbol coding is impossible, but the failure to find a master iterating matrix for $p = 2$, $q = 5$, among the eight matrices examined, suggests the interesting possibility that symbol coding may be proven impossible for this case, which is known to have a close-packed code (see *Note*).

Consider now the extension of the foregoing coding matrix construction to the cases in which the symbol order is the square ($q = 2$) of any odd prime. We may, without loss of generality, postulate the following matrix:

$$\begin{array}{cc} p_1 & ap_2 \\ p_2 & -bp_1 \end{array} \tag{15}$$

where $p_1 = \begin{matrix}1\\0\end{matrix}$, $p_2 = \begin{matrix}0\\1\end{matrix}$. In order to determine under what conditions matrix (15) is a master iteration matrix, condition (13) is applied to (15), and we obtain the requirements

$$\begin{matrix} & \alpha_1\beta_1 - \alpha_2\beta_2 b \not\equiv 0 \\ \text{or} & \alpha_1\beta_2 a + \alpha_2\beta_1 \not\equiv 0 \end{matrix} \quad (\text{mod. } p) \tag{16}$$

or both.

If we consider the β's as the variables with respect to which the set of homogeneous congruences of (16) must be unsolvable, condition (16) can be replaced by the equivalent necessary and sufficient condition that the determinant of the coefficients of the β's do not vanish:

$$D = \begin{vmatrix} \alpha_1 & -b\alpha_2 \\ \alpha_2 & a\alpha_1 \end{vmatrix} = a\alpha_1^2 + b\alpha_2^2 \not\equiv 0 \quad (\text{mod. } p). \tag{17}$$

It is obvious that condition (17) is satisfied whenever

$$a \not\equiv 0, \quad b \not\equiv 0,$$

and

$$\alpha_1 \not\equiv 0, \quad \alpha_2 \equiv 0, \quad (\text{mod. } p)$$

or

$$\alpha_1 \equiv 0, \quad \alpha_2 \not\equiv 0.$$

and we examine now the cases in which

$$\begin{matrix} a \not\equiv 0, & b \not\equiv 0, \\ \alpha_1 \not\equiv 0, & \alpha_2 \not\equiv 0. \end{matrix} \quad (\text{mod. } p)$$

It is known[6] that when $p \equiv 3 \pmod{4}$ p cannot divide numbers of the form $x^2 + 1$. Therefore, if we make $a = b = 1$ in (17) and set $\alpha_1 \equiv \alpha_2 x$, we obtain

$$D = \alpha_1^2 + \alpha_2^2 \equiv \alpha_2^2(x^2 + 1) \not\equiv 0. \ (\text{mod.}\ p = 4n + 3)$$

When we make $a = 1$, $b = 2$ in (17), we obtain a quadratic expression of the form $\alpha_1^2 + 2\,\alpha_2^2$, and since numbers of the form $x^2 + 2$ are not divisible by primes[7] $\equiv 5 \pmod{8}$, we obtain, setting $\alpha_1 \equiv \alpha_2 x$:

$$D = \alpha_1^2 + 2\alpha_2^2 \equiv \alpha_2^2(x^2 + 2) \not\equiv 0. \quad (\text{mod.}\ p = 8n + 5)$$

Finally, when we make $a = 2$, $b = 3$ in (17), we obtain an expression of the form $2\alpha_1^2 + 3\alpha_2^2$, and since numbers of the form $2x^2 + 3$ cannot be divided by primes[8] of the form $24n + 1$ or $24n + 17$, we obtain, setting $\alpha_1 \equiv \alpha_2 x$:

$$D = 2\alpha_1^2 + 3\alpha_2^2 \equiv \alpha_2^2(2x^2 + 3) \not\equiv 0.$$

$$(\text{mod.}\ p = 24n + 1, 24n + 17).$$

It is seen readily that the last two cases cover all primes of the form $4n + 1$, and since the case $a = b = 1$, covers all primes of the form $4n + 3$, all odd primes are covered. Therefore, master iteration matrices can be constructed for all squares of odd primes.

As an example, consider the case $p = 5$. Condition (17) is satisfied when $a = 1$, $b = 2$, and the matrix

$$\begin{matrix} v_1 & v_2 \\ v_2 & -2v_1 \end{matrix} \tag{18}$$

is a master iteration matrix modulo 5. On the other hand, matrix (11) is not a master iteration matrix modulo 5, because (12) vanishes modulo 5 when $\alpha_1 = \beta_2 = 1$, $\alpha_2 = 3$, $\beta_1 = 2$, and when the p_{ij}'s are the b's of (11).

The question of whether the search for master iteration matrices for values of q higher than 2 can be systematized is proposed as an interesting unsolved problem of coding or group theory.

Miscellaneous Notes

An interesting condition has been utilized by Zaremba[9] for proving the nonexistence of certain codes. This condition will be described in connection with its application to the once surmised close-packed code for correcting up to two errors in 90-binary symbol messages. These numbers satisfy the condition previously published[1] that the first three numbers of the corresponding line of Pascal's triangle add up to a power of 2:

$$1 + 90 + \frac{90.89}{2} = 2^{12}.$$

If such a code were to exist, the code book would contain 2^{78} code messages with the minimum distance 5, and with the allowable assumption that the all-zero message belongs to the code book, the remaining code messages with the greatest number of 0's would contain 85 0's and 5 1's. For the code to be close packed, every message with 87 0's and 3 1's should be obtainable in only one way by replacing 2 0's by 1's in the 85–0 and 5–1 code messages, and this condition determines the number of the 85–0 and 5–1 code messages. A single error in the 0's of these would produce 86–0 and 4–1 messages, and the number of remaining 86–0 and 4–1 messages determines the number of 84–0 and 6–1 code messages. This procedure can be extended to determine all the numbers of code messages with an increasing number of 1's. The numbers thus found should be integers, and this necessary but not sufficient condition for the existence of a close packed code constitutes Zaremba's condition. In the particular instance examined here, it is found that the number of 83–0 and 7–1 code messages is fractional, and this demonstrates anew the impossibility of the close-packed code surmised above.

When Zaremba's condition is applied to the singular 3-error correcting code with 12 binary information symbols and 11 check symbols built with the matrix previously published, it is found that there are: 1, 253, 506, 1288, 1288, 506, 253 and 1 messages with 0, 7, 8, 11, 12, 15, 16 and 23 1's, respectively.

Of course, the addition of a check digit serves to cluster the code messages even more into 1, 759, 2576, 759 and 1 message with 0, 8, 12, 16 and 24 1's, respectively.

The application of Zaremba's condition to the other singular 2-error correcting close packed code with 6 information and 5 check ternary symbols shows that there are 1, 132, 132, 330, 110 and 24 code messages with 11, 6, 5, 3, 2 and no 0's, respectively.

The application of Zaremba's condition to the hypothetical 1 error correcting code with 5 information and and 2 check symbols of order 6 does not disprove the existence of this code, for we find that the code book should contain 1, 175, 525, 1890, 3010 and 2175 code messages having, respectively, 7, 4, 3, 2, 1 and no 0's.

It is not known at present whether this code exists or not, but if it exists, it can be shown not to have the following property exhibited by the (2, 2, 2) code tabulated earlier.

Consider any code message, and leaving unchanged any 0's it may have, add respectively 1 and 2 modulo 3 to all other symbols, in the number system of order 3 with the symbols 1, 2 and 3. The two new messages thus obtained form part of the code (for instance, from the code message 01323 we derive the other code message 02131 and 03212).

Consider now the hypothetical (2, 6, 1) code, and select the greatest number of code messages in which three given symbols are 0's. The most favorable assumption, for ease of code message "packing," is that this number be the same for all similar groups, and be equally broken down into code messages in which the other four positions are filled by three or four nonzero symbols. We obtain thus $525/35 = 15$ messages with no 0's for

[6] T. Nagell, "Introduction to Number Theory," John Wiley and Sons, Inc., New York, N. Y., p. 135; 1951.
[7] *Ibid.*, p. 136.
[8] *Ibid.*, p. 190.
[9] Private communication.

instance, in the first four positions, and $175/35 = 5$ messages each with a single 0 in any one of these first four positions, with 0's in the last three positions for all.

Postulate now that to each message corresponds 4 additional messages obtained by adding 1, 2, 3 and 4 modulo 5 to the nonzero symbols, in the number system of order 5 in which the symbols are 1, 2, 3, 4 and 5. We may assume, without loss of generality, that the code messages selected by this process are:

$$\begin{array}{ccccccc} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 & 0 & 0 & 0 \\ 1 & 3 & x & x & 0 & 0 & 0 \\ 1 & 4 & x & 0 & 0 & 0 & 0 \\ 1 & 5 & 0 & 3 & 0 & 0 & 0 \\ 1 & 0 & x & x & 0 & 0 & 0 \\ 0 & 1 & x & x & 0 & 0 & 0 \end{array}$$

and the 28 other messages obtained by adding 1, 2, 3 and 4 modulo 5 to the nonzero symbols. If the code exists, it should be possible to replace the x's by numbers which, for the whole rectangle, satisfy the condition that no four nonzero numbers exist at the apex of any rectangle, which have the property that the difference (mod. 5) of any two on a side is equal to the difference of the other two. This condition is imposed by the requirement that the minimum distance 3 exists between any two for the 35 code messages involved. It has been verified by inspection that this condition cannot be realized. Hence, if the (2, 6, 1) code exists, it cannot have the property postulated above.

List of Symbols

p = prime number base.

q = power of p in coding system of order p^q.

n = number of redundant digits in system of order p^q.

(n, p, q) = characterization of lossless code or coding matrix in system base p^q with n redundant digits.

X_m = redundant check digits in system of order p.

Y_k = information digits in system of order p.

E_m = check digits calculated after reception in system of order p.

X_{mi} = redundant check digits in system of order p^q.

Y_{kj} = information digits in system of order p^q.

E_{mi} = check digits calculated after reception in system of order p^q.

p_{ij} = q-dimensional term in matrices (whether master iteration matrices or not).

$b_1, b_2, \cdots, b_q$ = base elements $\begin{matrix} 1 & 0 & & 0 \\ 0 & 1 & & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & & 1 \end{matrix}$ in system of order 2^q.

$t_1, t_2, \cdots, t_q$ = same base elements in system of order 3^q.

$v_1, v_2, \cdots, v_q$ = same base elements in system of order 5^q.

$p_1, p_2, \cdots, p_q$ = same base elements in system of order p^q.

α_i and β_i = one-dimensional factors in system of order p.

Definition of Terms

Close-packed code: An e-error correcting code in which the number of words is given exactly by the expression:

$$\frac{q^{qn}}{\sum_{j=0}^{j=e} (p^q - 1)^j \binom{\frac{p^{qn} - 1}{p^q - 1}}{j}}.$$

Sum (p_1p_2 of ensembles p_1 and p_2): The ensemble formed by the addition modulo p of the homologous symbols of the individual ensembles. The symbol $\oplus$, sometimes used to designate such sums, is not used here in order to conserve space, as no confusion results from this convention.

Coding matrix: The matrix formed by the coefficients of (4) or (5).

Iteration matrix: One of p^q q-dimensional square matrices which are required to pass from the (n, p, q) coding matrix to the $(n + 1, p, q)$ coding matrix.

Master iterating matrix: A q-dimensional square matrix composed of q-dimensional elements, from which the p^q iteration matrices can be derived systematically, by the operation $\sum_{i=1}^{i=q} \alpha_i p_{ij}$.

Corrector: The ensemble of nq digits of order p obtained as a result of the checking operations defined by (4) and (5).

Base characteristic: The ensemble of nq digits of order p forming any column of a coding matrix below an X_{mi} or a Y_{kj}, (also simply the "characteristic" when $p = 2$, $q = 1$).

Derived characteristic: Any sum of the base characteristics below X_{mi}'s or Y_{kj}'s (i or $j = 1, 2, \cdots q$), with one of the weights $0, 1, \cdots, p - 1$ given to each base characteristic.

Note: After the submission of this article, Prof. F. L. Dennis discovered and communicated to the writer an MIM for the case $p = 2$, $q = 5$. Like the matrices listed in Table I, Dennis' matrix is symmetric with respect to the main diagonal, and its successive columns are shifted upward one element with respect to the preceding columns. Thus, Dennis' matrix is fully characterized by its first and fifth lines, the indices of which are:

1	2	3	4	5
5	13	24	35	134

On Sampling the Zeros of Bandwidth Limited Signals*

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Summary—The sampling theorem enables a band-limited signal to be expressed in terms of a set of sample point values, which occur at the Nyquist rate. The sampling theorem has been generalized to include nonuniform sampling and the use of derivatives of the signal. In the present paper, a sampling theorem has been developed which utilizes information related to the zeros of the signal. The concept of complex zeros is introduced to show that the zeros occur at the Nyquist rate. This sampling theorem can be of use for enabling the transmission of binary signals, such as facsimile and infinitely-clipped speech, over a continuous band-limited channel. The result indicates the desirability of developing a completely general theory of sampling applicable to the various situations which may arise in practice.

Role of Sampling in Communications

Modern communication theory has presented the fundamental requirement for matching the information source to the communication channel to achieve optimum efficiency in transmission. Although the "matching" is usually assumed to pertain to the statistical constraints of the message, it also may be interpreted in a broader sense to include the type of signal structure. Since the channel can be either continuous or discrete, as is true for the information source, the need for conversion from each form to the other often arises in many practical situations. Examples of each type include: the amplitude sampling of speech at uniform intervals prior to transmission over a microwave relay system employing pulse position modulation and the transmission of on-off or polar telegraph signals over a cable system. These types of conversion are accomplished by the sampling theorem,[1] which states that a function of time $f(t)$ which is band-limited in $(0, W)$ is uniquely determined by a knowledge of its ordinates at uniform intervals $1/2W$ seconds apart. Furthermore, for any arbitrary set of such sample points, a continuous function exists, band-limited in $(0, W)$ and passing through all points. The theorem may be expressed as

$$f(t) = \sum_{n=-\infty}^{\infty} f\left(\frac{n}{2W}\right) \frac{\sin \pi(2Wt - n)}{\pi(2Wt - n)}. \tag{1}$$

Applications of the theorem often involve sampling over a finite interval of time T. Consideration of only a finite number of samples causes an interpolation error which can be quite large at the edge of the interval. This error inside the interval can be reduced as the $2WT$ product is made large.

Other types of communication systems must deal with discrete signals (either at the source or in the transmission channel) occurring at irregular intervals. An example is the application of nonsynchronous time division multiplexing of speech channels.[2] This implies amplitude sampling at nonuniform but arbitrarily specified intervals, with theoretical justification provided[3-5] by the use of the Lagrange interpolation functions expressed as

$$f(t) = \sum_{n=-\infty}^{\infty} f(t_n) L_n(t) \tag{2}$$

where

$$L_n(t) = \frac{g(t)}{(t - t_n) g'(t_n)}$$

and

$$g(t) = \prod_{n=-\infty}^{\infty} \left(1 - \frac{t}{t_n}\right).$$

The instants of sampling t_n are arbitrary except that the average spacing between successive instants is $1/2\ W$.

Yen[5] has shown that the interpolation functions, $L_n(t)$, are band-limited so long as the number of nonuniform intervals are finite. Paley and Wiener[3] prove the validity of the expansion with an infinite number of nonuniform sampling intervals but with strong restrictions in the location of the sampling points. Thus, it is seen that the expansion expressed by (1) is actually a special case of the Lagrange interpolation formula.

Still other applications require the measurement of two or more quantities (for example, amplitude and slope) at sampling points separated by two or more Nyquist intervals. For this application, Jagerman and Fogel[6] have shown that

$$f(t) = \sum_{n=-\infty}^{\infty} \left[f\left(\frac{n}{W}\right) + \left(t - \frac{n}{W}\right) f'\left(\frac{n}{W}\right)\right] \left[\frac{\sin \pi(Wt - n)}{\pi(Wt - n)}\right]^2. \tag{3}$$

It is to be noted in all cases that the average rate of sampling cannot be less than $2W$ per second (the Nyquist rate).

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[4] N. Levenson, "Gap and Density Theorems," American Mathematical Society Colloquium Publication, New York, N. Y., vol. 26; 1940.

[5] J. L. Yen, "On the non-uniform sampling of bandwidth-limited signals," IRE Trans. on Circuit Theory, vol. CT-3, pp. 251–257; December, 1956.

[6] D. L. Jagerman and L. Fogel, "Some general aspects of the sampling theorem," IRE Trans. on Instrumentation Theory, vol. IT-2, pp. 139–146; December, 1956.

The Need for Implicit Sampling

All of the sampling procedures described above are associated with a set of sampling points t_n which are independent of $f(t)$. Some communication problems require the sampling points t_n to be determined by the characteristics of $f(t)$. An example of such a case is prompted by the discovery[7] of the relatively small effect of infinite clipping on speech intelligibility. This suggests the possibility of transmitting a continuous speech signal $f(t)$ over a discrete channel by preserving the occurrence of the zero crossings and nothing else. The reverse situation occurs when it is desired to transmit black and white facsimile over a continuous channel. Here, the information source is a discrete (binary) signal with abrupt zero crossings at random points in time. Maximum conservation of bandwidth may be achieved by generating a band-limited signal which crosses the zero axis in step with the binary signal. After transmission over the continuous channel, the source signal could easily be reconstituted by infinite clipping, although to obtain this minimum possible transmission bandwidth, the continuous channel may be required to handle much greater peak power than normal.

Generalized Definition of Zero Crossings

The need is thus evident for a sampling theorem which deals with the location of the zero crossings, rather than the amplitudes (or slopes) at specified instants. A significant characteristic to be considered is the average rate of the zero crossings. In general, it is less than the Nyquist rate. For example, consider the case of a random-noise signal $n(t)$ with a flat spectral density band-limited in $(0, W)$. Rice[8] has shown that such a noise signal will cross the zero axis at an average rate of $2W\sqrt{\frac{1}{3}}$ per second. He has also shown that for the mth time derivative of $n(t)$, the average rate of zero crossings will be

$$2W\sqrt{\frac{2m+1}{2m+3}} \text{ per second,}$$

thus indicating that differentiating an infinite number of times would be required to obtain a signal with zeros occurring at the Nyquist rate. The zeros of a high-order derivative of $n(t)$, however, are very closely correlated, and no longer represent independent samples.

The deficiency in zero crossings is explained in a set of theorems by Titchmarsh,[9] the interpretation of which can be summarized as follows. Let $f(t)$ be band-limited in $(0, W)$, and let $V(f)$ be its (double-ended) Fourier transform.[10] Now consider the complex function

[7] J. C. R. Licklider and I. Pollack, "Effects of differentiation, integration, and peak clipping upon the intelligibility of speach," *J. Acoust. Soc. Amer.*, vol. 20, pp. 42–51; January, 1958.

[8] S. O. Rice, "Mathematical analysis of random noise," *Bell Sys. Tech. J.*, vol. 24, p. 54; January, 1945.

[9] E. C. Titchmarsh, "The zeros of certain integral functions," *Proc. Lond. Math. Soc.*, vol. 25, pp. 283–302; May 14, 1926.

[10] Strictly speaking an infinite sample of a signal like random noise does not possess a Fourier transform. However, the results established in this paper can be applied to such functions by a suitable limiting process.

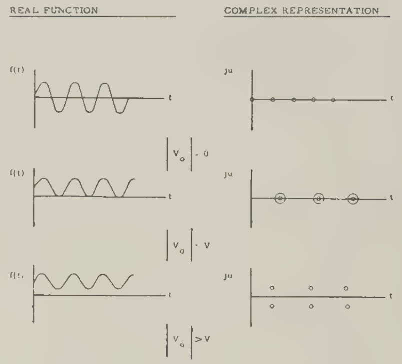

Fig. 1—Real and complex representation of $V_0 + V \sin 2\pi$ Wt.

$$f(z) = \int_{-W}^{W} V(f)e^{2\pi jfz}\, df$$

where $z = t + ju$ is a complex variable whose real axis coincides with the real time axis. [*Note*: $f(z)$ along the real axis coincides with $f(t)$.] Then $f(z)$ is a real, entire function, described by the location of its zeros, which are either real or occur in complex conjugate pairs. In general, the complex zeros tend to cluster near the real axis. Furthermore, the aggregate of zeros (including real and complex ones) occur, in the limit, at the Nyquist rate. The function thus is expressible by the infinite product

$$f(z) = f(0) \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) \tag{4}$$

where $f(0) \neq 0$, $z_n = R_n e^{j\theta_n}$ is the location of the nth zero,

$$R_n \leq R_{n+1}$$

and

$$\lim_{n\to\infty} \frac{2WR_n}{n} = 1.$$

It is hence apparent that continuous band-limited functions generated by natural information sources will include "complex" zeros which, unlike real zeros, are not physically detectable. A simple example of complex zeros can be given with a signal structure which has no information content but is useful in illustrating the nature of the zeros. Consider a single frequency sine wave combined with a dc bias voltage

$$f(t) = V_0 + V \sin 2\pi Wt.$$

For $V_0 = 0$ the real zeros of $f(t)$ occur at regular intervals $1/2W$ seconds apart. As V_0 increases, the zeros of $f(t)$ will start to converge in pairs, ultimately forming second-order zeros when $|V_0| = V$. (See Fig. 1.) If $|V_0| > V$, then $f(t)$ will never vanish. However, if t is generalized to a complex argument $z = t + ju$, then it will be found that $f(z)$ admits a uniform distribution of complex zeros located at

$$\left(n + \frac{1}{2}\right)\frac{1}{2W} \pm j \cosh^{-1}\left|\frac{V_0}{V}\right|$$

where n includes all odd or all even integers depending on whether V_0/V is positive or negative. Note that the larger dc biasing signal will move the complex zeros further from the real time axis. For a more general signal band-limited in $(0, W)$, the distribution of complex zeros will be irregular (except for symmetry about the real time axis) and as yet no procedure has been formulated for locating them from the knowledge of the continuous function.

Synthesis of a Band-Limited Function with a Given Set of Zeros

The preceding discussion has indicated that a band-limited function can be expressed by (4) in terms of its real and complex zeros which occur in the limit at the Nyquist rate. The formula, however, requires the knowledge of all past and future zeros, both real and complex, to reproduce $f(t)$. A more practical problem is that of representing a band-limited function over a finite interval in terms of the zeros which occur within this interval. If the contributions of the zeros outside of the specified interval are ignored, the resulting finite continued product will be a high-order polynomial and will have gross distortion near the edges of interval. An exponential correction of the amplitude has been discussed[11] based on the assumption that the zeros outside of the interval of interest are real and occur on the average at the Nyquist rate. This assumption is justified by the theoretical result that the zeros tend to cluster along the real axis for any band-limited function.

Another representation, which can be more easily related to the sampling theorem (1), is obtained by making the reasonable approximation that outside of the specified interval the zeros are all real and occur exactly at the Nyquist rate, while inside the interval they occur at slightly less than the Nyquist rate. As with (4), the function, for simplicity, is required to be nonzero at $t = 0$. Let the interval in question extend from $-T/2$ to $T/2$, and let N be the largest integer not exceeding TW. Outside of the interval, the zeros are assumed to occur at times equal to $\pm n/2W$, for $n = N + 1, N + 2, \cdots \infty$, and inside there are a maximum of $2N$ real or complex zeros $z_n = t_n + ju_n$, satisfying $|t_n| — T/2$. From (4), the band-limited function may be expressed as

$$f(t) = f(0) \prod_n (1 - t/z_n) \prod_{n=N+1}^{\infty} \left[1 - \left(\frac{2Wt}{n}\right)^2\right]$$
$$= f(0) \frac{\prod_n (1 - t/z_n)}{2\pi Wt \prod_{n=1}^{N} \left[1 - \left(\frac{2Wt}{n}\right)^2\right]} \sin 2\pi Wt \qquad (5)$$

in which is used the well-known infinite-product representation of the sine. The product in the numerator is extended over values of n corresponding to the zeros z_n within the interval.

[11] F. E. Bond, "The information-bearing elements in bandwidth limited signals," Doctoral dissertation submitted to Polytechnic Institute of Brooklyn; June, 1956.

With the limitation on the number of zeros of $f(t)$ within the interval in question, the factor multiplying $\sin 2\pi Wt$ in (5) is a proper rational fraction and may be expressed as the partial fraction expansion

$$f(0) \frac{\prod_n (1 - t/z_n)}{2\pi Wt \prod_{n=1}^{N} \left(1 + \frac{2Wt}{n}\right)\left(1 - \frac{2Wt}{n}\right)} = \sum_{n=-N}^{N} \frac{A_n}{2\pi Wt - \pi n} \qquad (6)$$

where

$$A_n = f(0) \frac{\prod_m \frac{1}{2Wz_m} \prod_m (2Wz_m - n)}{\prod_{m=1}^{N} - \left(\frac{1}{m}\right)^2 \prod_{\substack{m=-N \\ m\neq n}}^{N} (m - n)}. \qquad (7)$$

From (5) and (6), the result is obtained that

$$f(t) = \sum_{n=-N}^{N} \frac{A_n \sin 2\pi Wt}{2\pi Wt - \pi n}$$
$$= \sum_{n=-N}^{N} (-1)^n A_n \frac{\sin (2\pi Wt - \pi n)}{2\pi Wt - \pi n} \qquad (8)$$

which is seen to have exactly the form of (1) with a finite number of terms. Thus, for a specified set of real and complex zeros within an interval, a band-limited function can be derived having those zeros within the interval and having real zeros exactly at the Nyquist rate outside the interval.

The amplitudes at the sampling points, given by (7), may be expressed in terms of the migrations of the zeros from the sample point locations, thereby giving a clearer interpretation of the result. Let the migration of the nth zero be given by

$$\Delta_n = t_n - n/2W$$
$$(= z_n - n/2W \text{ for complex zeros}). \qquad (9)$$

Note that for at least one value of n, no zero is associated with the sample point and no zero migration is defined. Disregarding the fixed scale factor, the amplitude of the nth sample point is

$$(-1)^n A_n = (\_1)^n \frac{\prod_m (2Wz_m - m + m - n)}{\prod_{\substack{m=-N \\ m\neq n}}^{N} (m - n)}$$
$$= (-1)^n \frac{2W\Delta_n}{\prod_{\substack{m \\ \text{for which} \\ \text{no zeros are} \\ \text{associated.} \\ m\neq n}} (m - n)} \prod_{\substack{m \\ \text{for which} \\ \text{zeros are} \\ \text{associated} \\ m\neq n}} \left(1 + \frac{2W\Delta_m}{m - n}\right) \qquad (10)$$

where the external $2W\Delta_n$ is replaced by unity if no zero is associated with the sample point. This result shows that the location of a zero (specifically the migration from a sample point location) affects the amplitude of $f(t)$ in its immediate vicinity but does not have a marked effect on the signal at a much earlier or later time. A large migration, resulting in a large interval between successive zeros, will produce a large signal amplitude.

As mentioned earlier, the complex zeros, in general, are not specified or known. For example, if a binary signal, such as infinitely-clipped speech, is to be replaced by a band-limited signal having the same zero crossings, only the real zeros are given. To assume that complex zeros exist in addition to the specified real zeros would increase the bandwidth of the resulting band-limited signal. Nevertheless, the complex zeros may be introduced to reduce the large signal amplitude which otherwise would result when there is a large spacing between successive real zeros. This is an example illustrating qualitatively how bandwidth may be exchanged for signal-to-noise ratio when information is to be transmitted.

A simple illustration of the theory will now be given. Suppose that the zero crossings of the binary signal within the interval T in Fig. 2 are to be transmitted. The dotted extension of the given signal shows how real zeros are assumed to occur at a uniform rate outside of the specified interval. Since there are four zeros in the interval, a bandwidth W is chosen so that TW is not less than $4/2 = 2$. The zero-crossing migrations from the sample points are

$$2W\Delta_{-2} = 0$$

$$2W\Delta_0 = +\tfrac{1}{4}$$

$$2W\Delta_1 = -\tfrac{1}{4}$$

$$2W\Delta_2 = 0$$

and no zero is associated with the sample point at -1. The calculated band-limited function having the specified zero crossings is shown in Fig. 3. As expected, the relatively long spacing between the zeros at -2 and 0.25 contributes to a large amplitude in this region, and a channel with a high peak power capacity is required to transmit the signal.

Conversion of "Migration" of Zeros to Amplitude Information

Although the Δ_n's have been shown to provide the unique definition of a band-limited signal (vanishing at $n/2W + \Delta_n$) by means of a straightforward formula, the complexity involved would tend to inhibit practical application. However, for the problem of transmitting the abrupt crossings of a discrete binary signal over a continuous channel, it would be relatively simple to convert the migration intervals Δ_n to pulse amplitudes occurring at uniform intervals $(1/2W)$ with the individual amplitudes proportional to Δ_n. In this case, $2W$ is the average number of crossings per second and the minimum theoretical transmission bandwidth remains W. The instrumentation could consist basically of linear-sweep generators which measure the intervals of time between the occurrences of the zero crossings and the sampling instants, as determined by the average rate of crossings (see Fig. 4). Conversion of the pulse amplitude sequence back to the original binary sequence with crossings at $n/2W + \Delta_n$ would be possible at the receiving end of the channel with similar instrumentation. Again, the dynamic range of the continuous signal would depend upon the distribution of intervals between crossings. If the nature of the discrete signal source were such that long intervals could occur between crossings, special precautions would be necessary to prevent the continuous signal from growing to a prohibitive large amplitude.

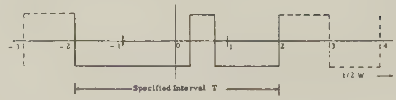

Fig. 2—Binary signal.

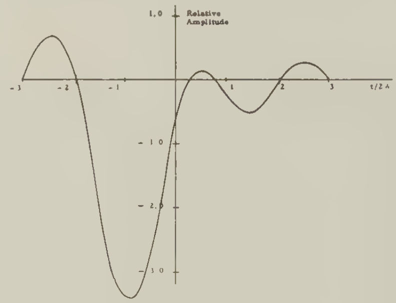

Fig. 3—Band-limited signal having the specified zero crossings.

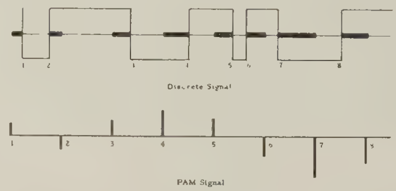

Fig. 4—Conversion of the migration of zero crossings of discrete signal to pulse amplitudes at uniform intervals

Need for a Generalized Theory of Sampling

It has been established that a signal band-limited in $(0, W)$ and essentially zero outside of an interval of duration T can be reproduced from a knowledge of $2WT$ samples (except in the vicinity of the ends of the interval where low accuracy may exist). The samples can include amplitudes at specified instants or, as is shown in this paper, the location (in the time domain) of the generalized zeros of the signal. Other theoretical methods of reproduction include the use of the $2WT$ Fourier Series components and the first $2WT$ derivatives at a single point. The types of sampling discussed here appear as logical requirements in certain practical communications problems, and are by no means intended to be exhaustive.

The various types of sampling may be viewed as aspects of a generalized theory of sampling, which is capable of covering all situations that may be encountered. Moreover, a generalized theory should be able to handle the practical case where the signals are not truly band-limited, since a sharp frequency cutoff is realizable only approximately with practical networks. Such a generalized theory has not yet been found, and a major breakthrough, comparable with the development of information theory itself, may be necessary.

Enhancement of Pulse Train Signals by Comb Filters*

JANIS GALEJS†

Summary—**The relative performance of different types of comb filters is investigated in conjunction with signal and noise types similar to those expected in radar applications. The filter types considered are idealized filters with zero transmission stop bands between their pass bands, optimum filters maximizing the peak signal-to-rms-noise ratio, cascaded delay line filters, feedback type filters, and storage tube filters. The pulse train signals consist of rectangular or sin x/x pulses with rectangular or sin x/x pulse envelope shapes. Power spectra of noise considered are rectangular and triangular. With a given number of signal pulses, the performances of the different filters vary from each other only by a few decibels in most cases analyzed. Storage tube filters exhibit lower signal-to-noise power ratios, but higher peak signal-to-rms-noise ratios, than the feedback type filters. Inaccurate delay times of filter delay lines are shown to decrease the peak signal output more than the signal power output and to affect the cascaded delay line filter less than the feedback type filter. Correlation techniques are compared with comb filters. The crosscorrelator exhibits the same peak signal-to-rms-noise ratio as the optimum filter.**

I. Introduction

COMB FILTERS for enhancing the detection of pulse train signals in additive noise and optimum filters which maximize the signal-to-noise ratio at a predetermined time instant have been the subjects of numerous investigations,[1-9] most of which consider one or a few filter types. The pulse-to-pulse integration, which characterizes the operation of comb filters, also takes place in storage tubes.[10,11] Correlation techniques[12,13] can be thought of as the counterpart of comb filters in time domain,[14] although their principles of operation are basically different.

An objective of this paper is to classify the relative improvements possible in detecting several types of pulse train signals in additive noise by various comb filters. The performance of theoretically optimum filters is compared with that of simpler and more practical filters. From the problems inherent in the design of comb filters only the effects of delay line tolerances are considered here. Also, the merits of autocorrelators and crosscorrelators relative to comb filters are determined.

II. General Considerations

In most analyses, the filter input is assumed to consist of rectangular pulses of constant amplitude. With a finite bandwidth and with phase distortions in the transmission system, the input pulses of the comb filter will be more or less distorted.[15] The rounded-off peak and the sloping sides of a heavily distorted pulse may be approximated by a sin x/x shaped pulse. The envelope of a pulse train received in a radar set depends on the antenna characteristics. The envelope can be approximated during the main lobe of the antenna pattern by a sin x/x curve as shown in Fig. 1. The signal output of the comb filter during the main lobe, which is of principal interest for computing the signal-to-noise improvement figures, is dependent only to a small degree on the signal during the side lobes. The spectrum of a pulse train with sin x/x pulses or pulse envelope has a rectangular spectrum envelope or rectangular spectral bands. This leads to expressions readily integrable in the subsequent analysis. The approximation of the actual pulse shapes or pulse envelopes by a sin x/x curve introduces a similar error in all the filter types analyzed, which is not expected to obscure the relative merits of the different filters.

Assuming a rectangular pass band of the IF amplifier, the noise of the radar set can be represented by band-limited white noise in the IF stages of the receiver. For small signal-to-noise ratios and for both linear and quadratic second detector characteristics, this noise spectrum is transformed into a triangular noise spectrum

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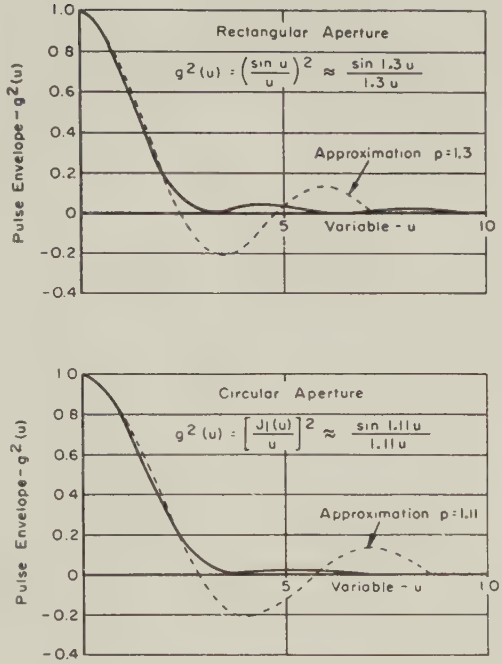

Fig. 1—Radiation patterns of parabolic reflector antennas—skin tracking.

after the second detector.[16] Although band-limited white noise represents the input noise of comb filters operating before the second detector, a triangular input noise spectrum should be considered for comb filters operating after the second detector.

Signal-to-noise ratios do not give a quantitative indication of the expected false alarm or incorrect dismissal probabilities, which serve as absolute criteria of the performance of a filter in conjunction with a threshold device. Nevertheless, signal-to-noise ratios can be used to determine the relative performance of different types of filters. The signal-to-noise ratios considered as criteria of filter performance here are

1) Signal power to noise power ratio,
2) Peak signal voltage to rms noise voltage ratio.

These ratios are defined in Appendix I to establish the nomenclature for the comb filter analysis.

III. Filters Having Physically Nonrealizable Comb Characteristics

The physically nonrealizable comb characteristics of the filters considered have infinite attenuation outside the pass bands. The pass bands of these filters are

1) Rectangular,
2) Sin x/x.

The pass band envelopes considered are

1) Rectangular,
2) Sin x/x,
3) Distorted sin x/x in an optimum filter approximation.

[16] *Ibid.*, pp. 158–161.

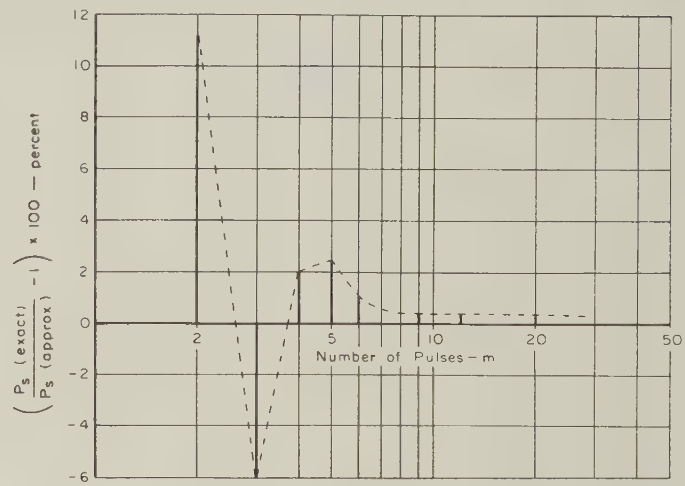

Fig. 2—Error of the approximate signal power representation.

Such filters have been investigated by George[4] for rectangular pulses and pulse envelopes, and for noise of flat spectrum. He neglected the energy overlap between adjacent pass bands.

The pass bands of filters are centered about the Pulse-Repetition Frequency (PRF) harmonics. The spread of the signal energy about the PRF harmonics is inversely proportional to the number of pulses in the pulse train. When the width of the pass bands is $2/mT$, where m is the number of pulses in the pulse train and T is the pulse repetition period, approximately 90 per cent of the signal energy about each PRF line is within the pass band of the filter. For a small number of pulses in the pulse train, the spectra about the PRF lines are relatively wide, and some of the energy of the signal about a PRF line will fall within the adjacent pass bands of the filter.[17] As seen from Fig. 2, the per cent error in the signal power computation resulting from the neglect of the sideband overlap is less than one per cent for the number of pulses, m, larger than six.

Values of the signal power to noise power improvement figures, M_p, of the filter types considered are listed in Table I (next page). Neglecting the overlap of the signal sidebands in the signal computations is equivalent to setting $f(m)$ of (9) equal to zero. Only slight changes occur in M_p by changing the input noise spectrum from triangular to rectangular. The M_p of the filters investigated for triangular noise differs by less than 1 db from the M_p values of the corresponding filters arrived at by George[4] in the case of rectangular noise.

IV. Filters Having Physically Realizable Comb Characteristics

The physically realizable filters have finite attenuation stop bands between the pass bands. These filters may be subdivided into

1) Optimum filters that maximize the instantaneous signal-to-rms-noise ratio,

[17] This partial overlap of signal energy is apparent from (20) of George and Zamanakos, *op. cit.*

TABLE I

Signal Power to Noise Power and Peak Signal Voltage to RMS Voltage Improvement Figures of Comb Filters

Pulse Shape	Pulse Train Envelope	Noise Power Spectrum		Filter Types								
				Physically Nonrealizable Comb Characteristics				Physically Realizable Comb Characteristics				
				rectangular pass bands rectangular envelope	rectangular pass bands sin x/x envelope	sin x/x pass bands sin x/x envelope	optimum for triangular noise	optimum for triangular noise	optimum for rectangular noise	cascaded delay line filter	feedback	feedback optimum for triangular noise
rectangular	rectangular	rectangular	$M_p - 10 \log 0.45\, m - f(m)$	0	2.1	4.25	3.8	3.46	3.80	1.70	1.84	3.59
			$M_v - 10 \log m$					2.48	2.56	1.36	0	1.14
		triangular	$M_p - 10 \log 0.45\, m - f(m)$	0	1.5	3.6	3.32	2.96	3.20	1.70	1.84	3.09
			$M_v - 10 \log m$					2.02	1.94	1.36	0	0.64
	sin x/x	rectangular	$M_p - 10 \log m_h$	0.5	2.62	2.62	2.43	3.92	4.29	0.06	−2.5	−0.75
			$M_v - 10 \log m_h$					1.58	1.66	0.42	−3.0	−1.86
		triangular	$M_p - 10 \log m_h$	0.5	2.0	2.0	1.68	3.44	3.69	0.06	−2.5	−1.25
			$M_v - 10 \log m_h$					1.12	1.04	0.42	−3.0	−2.36
sin x/x	rectangular	rectangular	$M_p - 10 \log 0.45\, m - f(m)$	0	1.93	4.08	2.1	1.76	1.70	1.70	1.84	1.89
			$M_v - 10 \log m$					−0.27	0	0	−1.36	−1.61
		triangular	$M_p - 10 \log 0.45\, m - f(m)$	0	1.33	3.43	2.47	2.11	1.70	1.70	1.84	2.24
			$M_v - 10 \log m$					0.12	0	0	−1.36	−1.24
	sin x/x	rectangular	$M_p - 10 \log m_h$	0.5	2.45	2.45	0.73	2.22	2.17	0.06	−2.5	−2.45
			$M_v - 10 \log m_h$					−1.17	−0.9	−0.95	−4.36	−4.61
		triangular	$M_p - 10 \log m_h$	0.5	1.83	1.83	0.83	2.59	2.17	0.06	−2.5	−2.10
			$M_v - 10 \log m_h$					−0.78	−0.9	−0.95	−4.36	−4.26

Note: For sin x/x pulse train envelopes, the physically realizable filters are designed to have the peak signal output at the half-power point after the peak of the pulse envelope.

2) Cascaded delay line filters,
3) Feedback type filters,
4) Feedback type filters modified to approximate the optimum filters,
5) Storage tube filters.

Characteristics of optimum filters that maximize the peak signal-to-rms-noise improvement figure, M_v, at a predetermined instant of time, t_0, have been arrived at by North,[1] Dwork,[2] and Zadeh and Ragazzini.[3] However, the signal-to-noise improvements of optimum filters relative to nonoptimum filters have not been explicitly stated. The transfer functions of the optimum filters considered in this paper are computed according to the method outlined by Zadeh and Ragazzini.[3] The resulting optimum filters consist of a cascade connection of a noise-shaping network, a single pulse filter, a non-feedback type comb filter, and an output delay line (see Figs. 3 and 4). The noise-shaping network has a transfer function equal to the reciprocal of the noise power spectrum, $P_N(\omega)$. The noise-shaping network tends to emphasize the spectral components of the signal where the noise spectrum has its lowest amplitudes. The single pulse filter changes the shape of the rectangular pulses into triangular pulses of double width. The single pulse filter is identical with North's optimum filter for a single rectangular pulse in white background noise. The comb filter sums the individual pulses of the pulse train where the pulses are weighted in proportion to their amplitudes. The delay time of the output delay line depends on the time, t_0, when M_v is maximized. This delay line does not affect the peak signal, signal power, or noise power outputs. The signal-to-noise improvement figures of optimum filters are of the form given by (9), (10), (18), and (19), with

$$f(m) = 10 \log (1 + 0.5 m^{-2}). \qquad (1)$$

The numerical values of the improvement figures are seen from Table I. For sin Ct/Ct pulse envelopes, only the optimum filters maximizing M_v at the peak of the pulse envelope or at the half power point following the peak of the pulse envelope are considered. In both cases, pulses starting with the half power point prior to the peak of

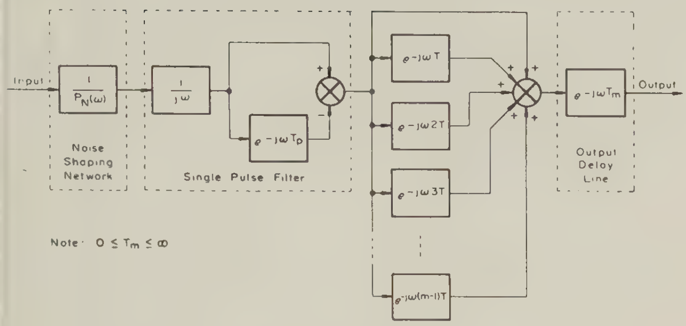

Fig. 3—Filter maximizing signal-to-noise ratio at or after the trailing edge of the last pulse of a uniform pulse train.

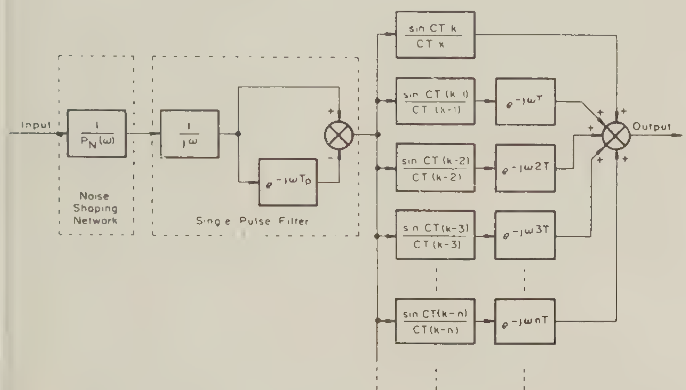

Fig. 4—Filter maximizing signal-to-noise ratio at the kth pulse after the peak of the (sin Ct/Ct) pulse envelope.

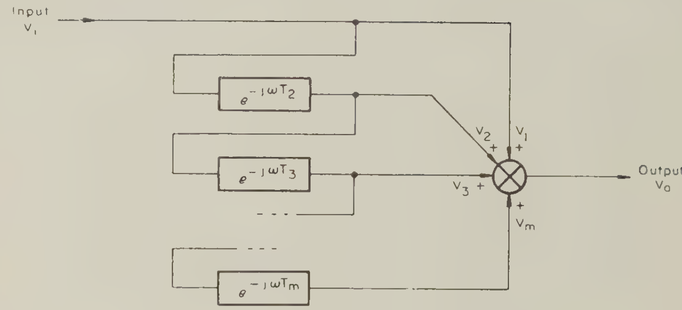

Fig. 5—Cascaded delay line comb filter.

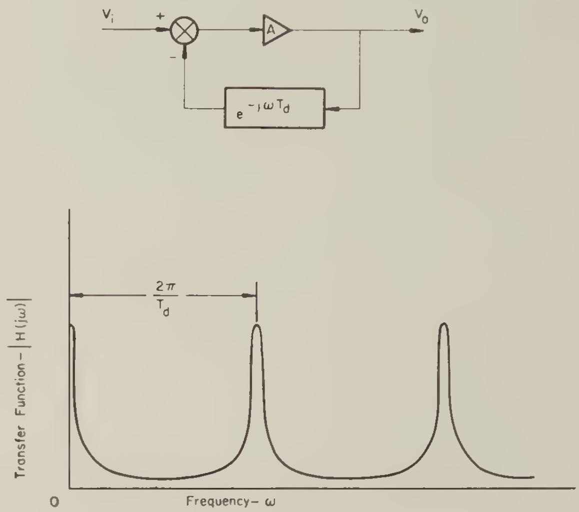

Fig. 6—A feedback type comb filter.

the pulse envelope contribute to the signal output at the instant when M_v is maximized. If the signal-to-noise ratio is maximized at the half power point of the signal envelope after the envelope peak instead of at the envelope peak, values of M_p are raised 4.8 db and values of M_v are raised 3 db. However, the higher signal-to-noise ratio requires twice the number of delay lines in the filter.

Cascaded delay line filters have been investigated by Bailey,[5] who compares them with feedback type filters. The cascaded delay line filters are equivalent to the optimum filters after the single pulse filter and the noise-shaping networks are removed (see Fig. 5). Since $f(m)$ of (9) is of the same form as for the optimum filters in (1), the signal-to-noise improvement figures of the cascaded delay line filters differ from the signal-to-noise improvement figures of the optimum filters only by the constants listed in Table I.

The simple feedback type filter consists of a delay line in a feedback path of an amplifier, as shown in Fig. 6. With the delay time equal to the pulse repetition period, the input pulses are added to the pulses circulating in the feedback loop. Stability considerations restrict the gain of the amplifier, A, to less than unity. Such filters have been analyzed in conjunction with rectangular pulses of constant amplitude.[5–7]

The signal-to-noise improvement figures, M_p and M_v, of the feedback type filter are listed in Appendix II. The peak signal-to-rms-noise improvement figures M_v are plotted in Figs. 7 and 8 for rectangular and (sin Ct/Ct) pulse envelopes, respectively. The signal-to-noise improvement figures, M_p and M_v, can be changed to the form of (9), (10), (18), and (19) after relating the amplifier gain to the number of pulses in the pulse train, as in (28) and (29). As seen from Table I, the signal-to-noise improvement of the feedback type filter is within 2.5 db of the optimum filters for uniform pulse trains, but within approximately 6 db for (sin Ct/Ct)-shaped pulse train envelopes.

The simple feedback type filter may be modified to approximate the characteristics of an optimum filter. It is cascaded with a single pulse filter and with a noise-shaping network. This makes the pass band envelope of the filter identical with the pass band envelope of the optimum filter. The individual pass bands, however, retain the shape of the pass bands of a simple feedback type filter. The resulting signal-to-noise improvement is seen in Table I.

There is a similarity between the data storage process in a storage tube and in a simple feedback type filter. It can be shown[10,11] that the storage of the charge distribution over one sweep decreases its amplitude by a factor $e^{-\gamma}$. A similar dependence of the amplitude of the stored pulses on storage time is observed in feedback type comb filters. Therefore, the integrating process of

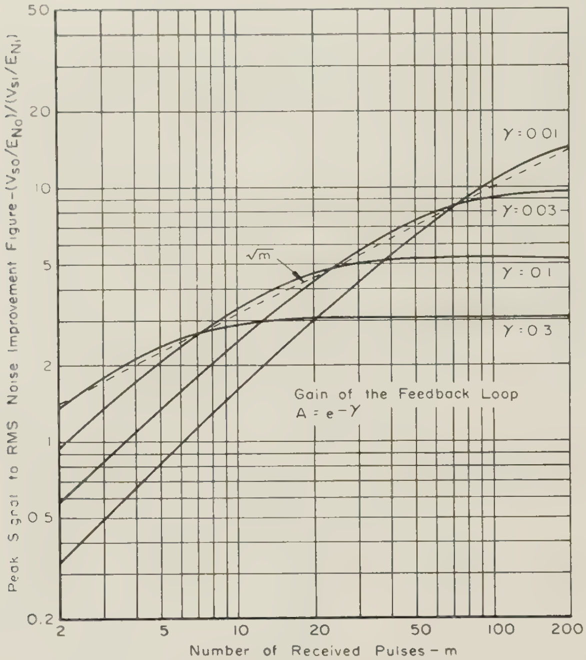

Fig. 7—Peak signal-to-rms-noise improvement figure of a simple feedback type comb filter.

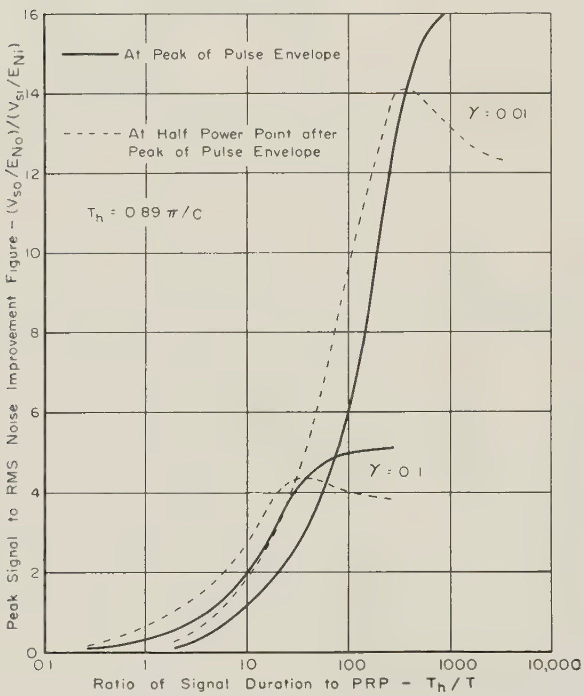

Fig. 8—Peak signal-to-rms-noise improvement figure for a simple feedback type comb filter with a pulse train having a (sin Ct/Ct) envelope.

the storage tube filter may be represented by a feedback loop having the open loop transfer function $e^{-\gamma}e^{-i\omega T}$. The storage tube accepts input information only during the write sweep and supplies output data only during the read sweeps. This gating operation can be represented by properly timed switches before and after the feedback loop, the action of which has been neglected in earlier analyses.[10,11] The equivalent representation of the storage tube thus obtained is shown in Fig. 9. The results of the storage tube analysis using the above representation are summarized in Appendix III.

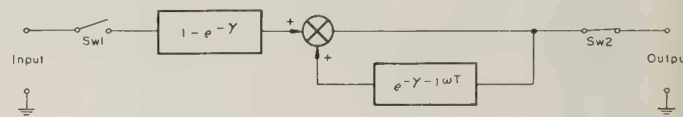

Fig. 9—Equivalent representation of a storage tube comb filter. Sw1–open during read sweeps; closed during write sweeps. Sw2–closed during read sweeps; open during write sweeps.

The signal power to noise power improvement figures, M_p, of the storage tube filter and of the simple feedback type filter are plotted in Figs. 10 and 11. The M_p figures of storage tube filters are generally lower than the M_p of feedback type filters, but the difference between these values does not exceed 1.4 db. The peak signal-to-rms-noise improvement, M_v, of storage tube filters has been plotted in Figs. 12 and 13, along with M_v of simple feedback type filters. Storage tube filters give approximately $\sqrt{k-1}$ times higher values of M_v than the simple feedback type comb filters, assuming one read sweep per k sweeps of the storage tube. This increase of the peak signal-to-rms-noise improvement figure is caused by the decrease of the rms noise output of the filter by the gating action of the storage tube. The peak signal-to-rms-noise improvement figure, M_v, of the storage tube filter can even exceed the M_v figure of the optimum filters when k is sufficiently large. However, there is almost no practical value in such a storage tube operation since it postulates an output device that is able to distinguish a signal peak from background noise having a low rms value, but high-peak-values in a decreased number of samples.

V. Discussion of the Tabulated Signal-to-Noise Improvement Figures

Signal-to-noise improvement figures of the filter types described in the two preceding sections are listed in Table I. The fundamental assumptions upon which the entries are based are as follows:

1) The rectangular signal pulses have a duration T_p.
2) The width of the sin x/x shaped pulses is T_p as measured between the nulls surrounding the central peak.
3) The idealized low-pass filter preceding the comb filter has a sharp cutoff at the frequency $\omega_p = 2\pi/T_p$.
4) The triangular noise power spectrum decreases to one half of its zero frequency value at the frequency ω_p. This is equivalent to assuming a predetector bandwidth of 2 ω_p.

Besides obtaining performance data on specific filters within the limitations of the above assumptions, the tabulated data help to formulate more general conclusions on the effects of the envelope or pulse shape.

The sin x/x pulse envelope was introduced to simulate the variable amplitude returns of a scanning radar (see Section II). It is seen that a change from rectangular to

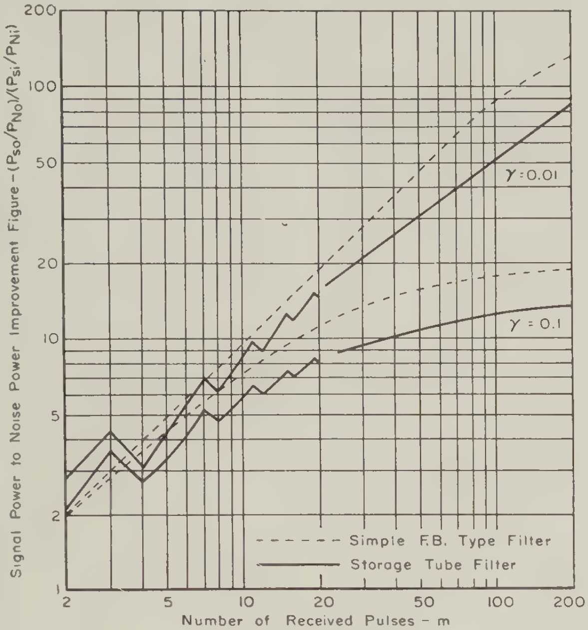

Fig. 10—Signal power to noise power improvement figure of storage tube and simple feedback type comb filters $k = 4$.

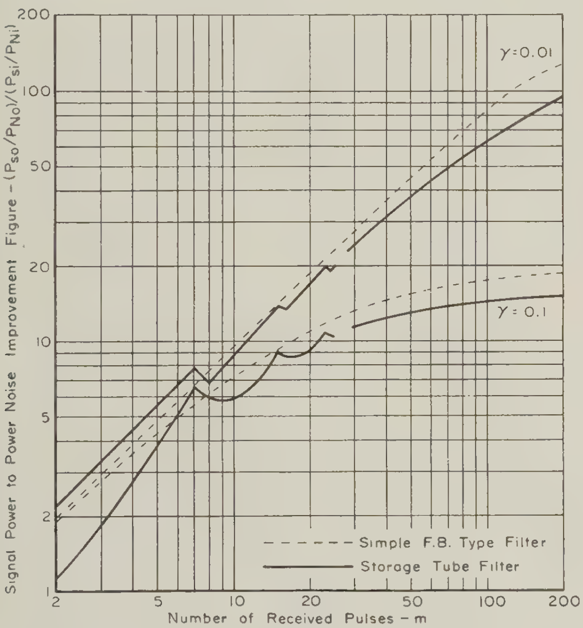

Fig. 11—Signal power to noise power improvement figure of storage tube and simple feedback type comb filters $k = 8$.

a sin x/x pulse envelope affects the feeback type filters most adversely. Their improvement figures M_p and M_v are decreased by 4.3 db and 3 db, respectively, if the number of pulses between the half-power points of the sin x/x pulse envelope m_h is set equal to the number of pulses in the rectangular pulse train m. The physically realizable optimum filters are affected less under similar conditions. Thus, M_v is decreased by only 0.9 db, but M_p is increased by 0.5 db.

The sin x/x pulse shape of the pulses is intended to simulate the effects of pulse distortion. Changing the pulses from rectangular to sin x/x shape decreases the

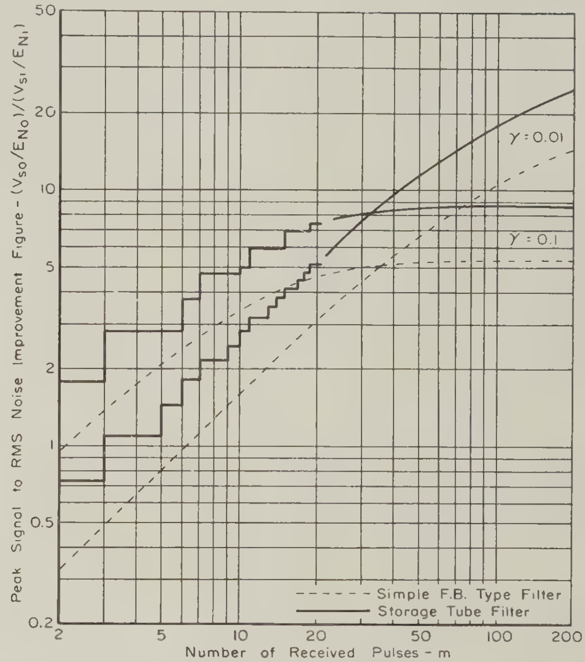

Fig. 12—Peak signal-to-rms-noise improvement figure of storage tube and simple feedback type comb filters $k = 4$.

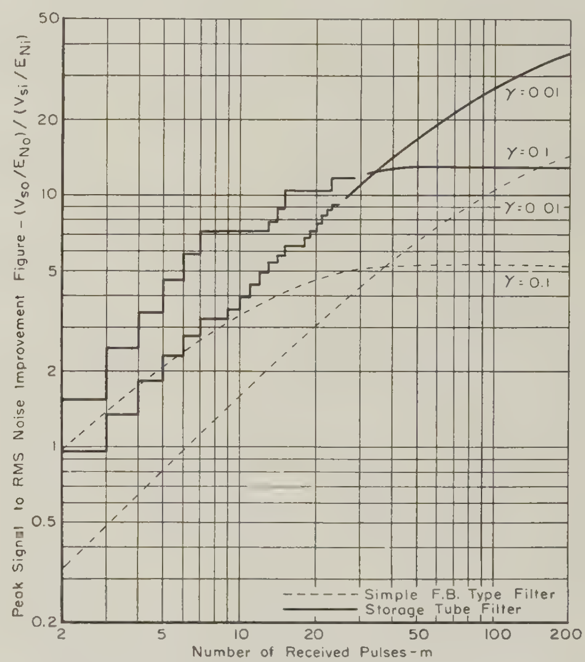

Fig. 13—Peak signal-to-rms-noise improvement figure of storage tube and simple feedback type comb filters $k = 8$.

improvement figures M_p and M_v of optimum physically realizable filters by less than 2.1 db and 2.7 db, respectively. The improvement figure M_p of the simple feedback type filter is not affected by a change of pulse shape, while M_v is decreased by approximately 1.3 db.

A general comparison between physically realizable filters and physically nonrealizable filters, which have zero transmission stop bands between their pass bands, can also be made with the aid of data of Table I. The improvement figures M_p of both filter types are of the

same order of magnitude. Therefore, no general improvements of filter performance can be expected by increasing the attenuation of filter stop bands. Tchebycheff filters of Graves,[8,9] which permit an arbitrary degree of attenuation in their stop bands, appear to have limited merits for filtering pulse train signals.

VI. Delay Time Inaccuracies of Comb Filters

Delay time inaccuracy is one of the factors which limits the performance of comb filters, particularly when integrating a large number of pulses. In the absence of experimental evidence about the structure of delay time errors, a normal delay time distribution appears to represent a model which will give at least a qualitative indication of the expected filter performance. Considering cascaded delay line and simple feedback type filters, it is assumed that the delay times of the cascaded delay line filter are normally distributed about the correct delay value, while the delay line in the simple feedback type filter is assumed to have normal variations of the delay time. The delay time error of a pulse will increase with its storage time in either one of the two filters. Considering the signal output of the filter at a specified time, each pulse of the train has a different probability P_i of contributing to the filter signal output. The average signal output $\overline{V_{so}}$ at this time is equal to the sum of all the probabilities P_i times the signal amplitude S. The delay time distribution or variation cause corresponding changes of the filter transfer function $H(j\omega)$. The signal output power P_{so} of the filter is given by an integral having $H(j\omega)$ in the integrand. The average $\overline{P_{so}}$ is thus the average of an integral. The procedure for computing the average values $\overline{V_{so}}$ and $\overline{P_{so}}$ is described for the feedback type filter in Appendix IV.

When computing the expected signal output of a single filter or the average signal output of a set of filters, the peak of the average signal $\overline{V_{so}}$ occurs at the center of the last pulse of the pulse train. The average signal output of the cascaded delay line filter during the last pulse of the train is plotted in Figs. 14 and 15 for $m = 11$, and 101, respectively. Larger values of the ratio between the standard deviation of the delay time and pulse duration T_σ/T_p cause the output to decrease in amplitude and to spread out in width. For $T_\sigma/T_p = \text{const} \neq 0$, the per cent drop-off of $\overline{V_{so}}$ increases with increasing m. This can also be seen from the plot of the filter average peak outputs $\overline{V_{so}}$ in Figs. 16 and 17.

The average signal power output $\overline{P_{so}}$ of the two filters is plotted in Figs. 18 and 19. $\overline{P_{so}}$ of the filters decreases less rapidly with decreasing delay time accuracy than $\overline{V_{so}}$. Thus, the peak signal output of a feedback type filter is decreased by 2 db when the number of pulses m_o is equal to the ratio of pulse duration to the standard deviation of the delay time. The signal power output of the same filter is decreased by 1.5 db at the same number of pulses, m_0. An inaccuracy of the delay time distorts the large amplitude output pulses made up of a large number of input pulses more than the lower amplitude

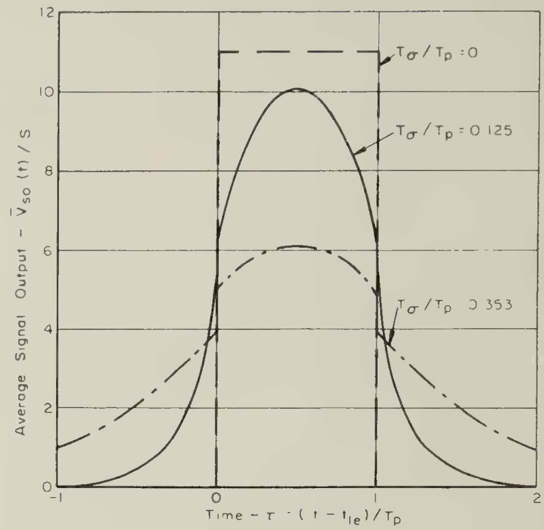

Fig. 14—Average signal output of a cascaded delay line filter $m = 11$.

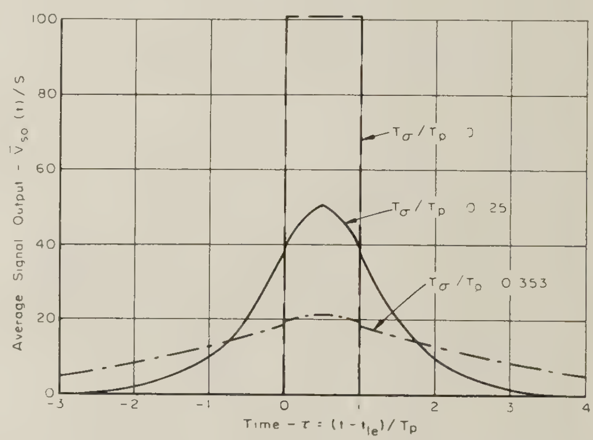

Fig. 15—Average signal output of a cascaded delay line filter $m = 101$.

output pulses. The output signal contributes during the buildup and during the decay of the pulse train to the signal power output. These output pulses are less distorted than the output pulses which occur during the peak signal output. The signal power output is therefore affected less by delay time inaccuracies than the peak signal output.

The noise output of the filter is not changed by delay time errors which are small compared with the pulse repetition period. Therefore, the signal-to-noise improvement figures of the filters decrease proportionally with the decrease of the average peak signal or of the average signal power output of the filter.

VII. Correlation Techniques

Correlation provides a means for improving the signal-to-noise ratio of pulse train signals. The correlator performs in time domain an operation similar to the operation of a comb filter in frequency domain. The comb filter passes harmonically related frequency components, while suppressing bands of frequencies lying between the passed frequency components. The correlator multiplies

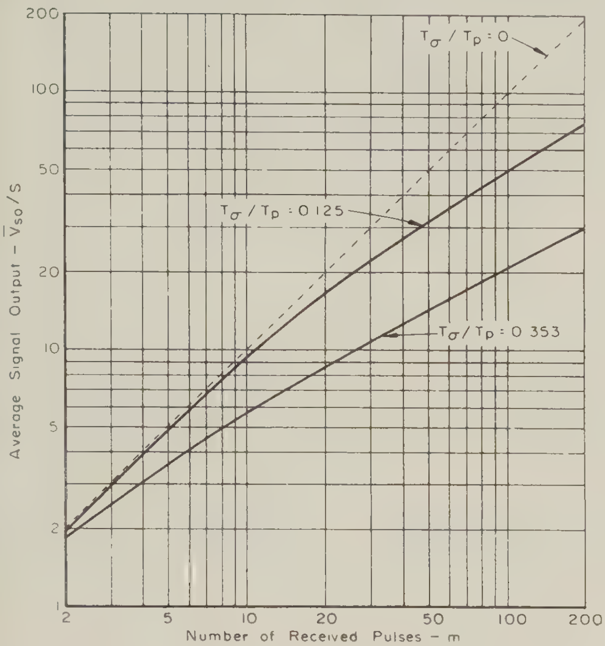

Fig. 16—Average peak signal output of a cascaded delay line filter.

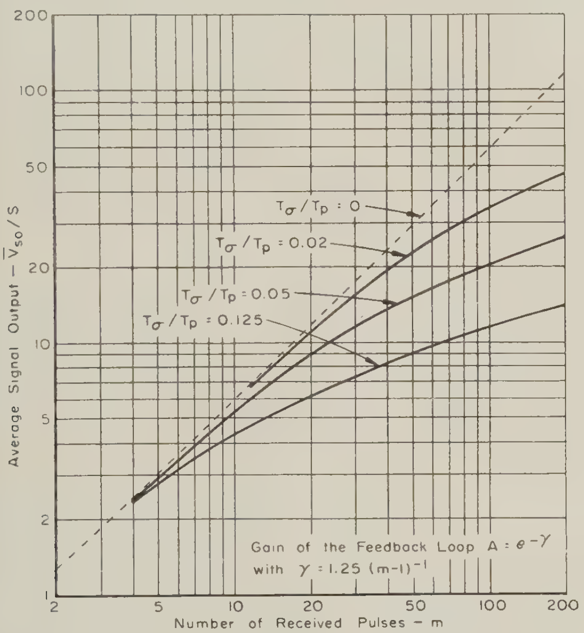

Fig. 17—Average peak signal output of a feedback type filter.

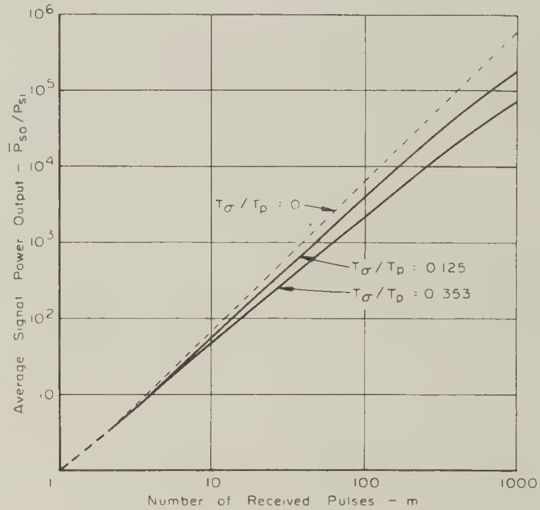

Fig. 18—Average signal power output of cascaded delay line filter.

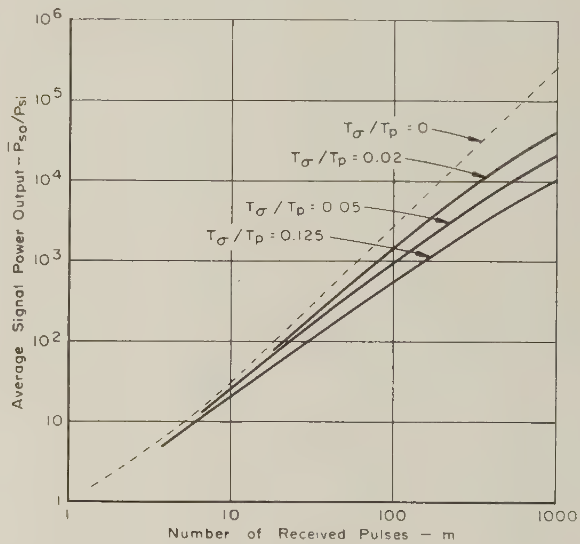

Fig. 19—Average signal power output of a simple feedback type filter.

the incoming signal with a reference signal and then averages the multiplier output in a low-pass filter. Only those frequency components present in both the input and the reference signal result in a zero frequency multiplier output component that passes through the low-pass filter. Lowering the cutoff frequency of the low-pass filter in the correlator is equivalent to narrowing the width of the comb filter pass bands.

The crosscorrelator multiplies the noise-perturbed input pulse train with a locally generated reference signal which is identical to the input signal with the noise removed. The multiplier output is integrated. The autocorrelator multiplies the input signal delayed by one pulse-repetition period with the undelayed input signal. The multiplier output is integrated. For minimum noise output, the integration starts when the first pulse appears in the delay line output and ends when the last pulse is applied to the delay line input. The signal-to-noise improvement figures of the correlators for pulse train signals can be derived by methods similar to the ones used for sinusoidal signals.[18] The development leading to the signal-to-noise improvement figure for the crosscorrelator is shown in Appendix V.

For large signal-to-noise ratios the autocorrelator type filter gives 3-db lower signal-to-noise ratios than the crosscorrelator.[13] For small input signal-to-noise ratios the output signal-to-noise ratio is quadratically dependent on the input signal-to-noise ratio of the autocorrelator. For small r_{vi} the autocorrelator is inferior to the cross-

[18] S. F. George, "Time Domain Correlation Detectors vs. Conventional Frequency Domain Detectors," Naval Res. Lab., Washington, D. C., Rep. 4332; May, 1954.

correlator, the output of which is always linearly dependent on the input signal-to-rms-noise ratio.[12,13]

The peak signal-to-rms-noise improvement, M_v, of the crosscorrelator is the same as M_v of the optimum filter.[14] The advantage of the simplicity of the correlator (which consists of a multiplier and an integrator) is offset by the required knowledge of the starting point of the received signal. If the starting point of the incoming signal is not accurately known, several correlator channels each having a different τ value must be provided. This increases the complexity of the equipment.

VIII. Conclusions

Comparison of several types of comb filters shows that with proper design their performance figures lie within a few decibels for a given number of signal pulses in the pulse train. The power spectrum of noise and the pulse or pulse envelope shapes are only of secondary significance. The feedback type comb filter, which is the simplest one to construct, gives signal-to-noise improvements within 2.5 db and 4–6 db of the improvements possible with optimum filters for rectangular and sin x/x shaped pulse envelopes, respectively. The storage tube filter gives higher peak signal-to-rms-noise ratios than the feedback type filter. However, it does not appear feasible to make practical use of this difference.

Inaccurate delay times of the filter delay lines affect the signal output of the filter. The peak signal output of the feedback type filter is decreased by 2 db when the number of pulses is equal to the ratio of pulse duration to the standard deviation of the delay time. The signal power output of the same filter is decreased by 1.5 db at the same number of pulses.

Crosscorrelators give the same peak signal-to-rms-noise output as the optimum filters.

Appendix I

Criteria of Filter Performance

The signal power at the filter input is given by

$$P_{si} = \frac{2\pi}{T_a} \int_{-\omega_p}^{\omega_p} | V_s^t(\omega) |^2 \, d\omega \tag{2}$$

where $V_s^t(\omega)$ is the signal spectrum, T_a is an averaging interval, $\omega_p = 2\pi/T_p$, and T_p is the pulse duration. For rectangular pulses this integration interval encompasses 90 per cent of the total signal power. For sin x/x shaped pulses it encompasses all the signal power. Although the selection of this interval is arbitrary, it is consistent with other investigations.[4] The noise power input is

$$P_{Ni} = \int_{-\omega_p}^{\omega_p} P_N(\omega) \, d\omega \tag{3}$$

where $P_N(\omega)$ is the power spectrum of noise. The output signal and noise powers are

$$P_{so} = \frac{2\pi}{T_a} \int_{-\omega_p}^{\omega_p} | V_s^t(\omega) |^2 \, | H(j\omega) |^2 \, d\omega \tag{4}$$

and

$$P_{No} = \int_{-\omega_p}^{\omega_p} | H(j\omega) |^2 \, P_N(\omega) \, d\omega. \tag{5}$$

Expressing the signal-to-noise ratio in decibels,

$$r_{pi} = 10 \log \frac{P_{si}}{P_{Ni}} \tag{6}$$

and

$$r_{po} = 10 \log \frac{P_{so}}{P_{No}}. \tag{7}$$

The power improvement figure of the filter is

$$M_p = r_{po} - r_{pi} \text{ (db)}. \tag{8}$$

The improvement figure is related to the number of pulses in the pulse train, m, or to the number of pulses between the half power points of the pulse envelope, m_h, as

$$M_p = 10 \log 0.45m + f(m) + c \tag{9}$$

$$M_p = 10 \log m_h + c. \tag{10}$$

The functions $f(m)$ and constants c depend on the particular filter type under consideration. The proportionality factor 0.45 in the first term of (9) was selected to provide a direct comparison with the results of George.[4]

The peak signal input V_{si} is equal to the signal amplitude S

$$V_{si} = S. \tag{11}$$

The peak signal output V_{so} is given by

$$V_{so} = \max | V_{so}(t) | = \max \left| \int_{-\omega_p}^{\omega_p} V_s^t(\omega) H(j\omega) e^{-j\omega t} \, d\omega \right|. \tag{12}$$

The rms noise input and output voltages are given by

$$E_{Ni} = \sqrt{P_{Ni}} \tag{13}$$

and

$$E_{No} = \sqrt{P_{No}} \tag{14}$$

assuming 1-ohm impedance levels. Expressing the signal-to-noise ratios in decibels,

$$r_{vi} = 20 \log \frac{V_{si}}{E_{Ni}} \tag{15}$$

and

$$r_{vo} = 20 \log \frac{V_{so}}{E_{No}}. \tag{16}$$

The voltage improvement figure of the filter is

$$M_v = r_{vo} - r_{vi} \text{ (db)}. \tag{17}$$

The improvement figure is related to the number of pulses in the pulse train, m, or to the number of pulses between the half power points of the pulse envelope, m_h, as

$$M_v = 10 \log m + c \tag{18}$$

$$M_v = 10 \log m_h + c. \tag{19}$$

The constants, c, depend on the particular filter type under consideration.

Appendix II

Signal-to-Noise Improvement Figures of the Feedback Type Filters

The signal-to-noise improvement figures of the feedback type filters are computed by substituting the filter transfer function and the signal and noise parameters in the relations listed in Appendix I.

For rectangular pulses of constant amplitude and of triangular or rectangular noise spectrum, M_p is given by[7]

$$M_p = 10 \log \left[\frac{1+A}{1-A} - \frac{2A}{(1-A)^2} \frac{1-A^m}{m} \right] \tag{20}$$

and M_v is given by[19]

$$M_v = 20 \log \left[1.17(1-A^m) \sqrt{\frac{1+A}{1-A}} \right] \tag{21}$$

which is plotted in Fig. 7. For rectangular pulses and a sin Ct/Ct envelope of the pulse train,

$$M_p = 10 \log \left[\frac{1}{TC} \tan^{-1} \left(\frac{1+A}{1-A} \frac{CT}{2} \right) \right] \tag{22}$$

$$M_v = 20 \log \left[1.17\beta \left(e^{-1.885\beta} M + \frac{\beta}{\beta^2 + 0.31} \right) \right] + 10 \log \frac{1+A}{1-A}, \tag{23}$$

at the time of the peak of signal envelope or

$$M_v = 20 \log \left[1.17\beta \left(e^{-3.295\beta} M + \frac{0.707\beta + 0.392}{\beta^2 + 0.31} \right) \right] + 10 \log \frac{1+A}{1-A} \tag{24}$$

at the time of the half power point after the peak of the pulse envelope, where

$$M = \frac{0.512\beta - 0.0054}{\beta^2 + 0.658\beta + 1.108} - \frac{0.506\beta - 0.480}{\beta^2 + 0.31} \tag{25}$$

$$\beta = \gamma / TC \tag{26}$$

$$A = e^{-\gamma}. \tag{27}$$

A plot of (23) and (24) is shown in Fig. 8. The improvement figure M_v is maximized for a uniform pulse train if

$$\gamma = 1.25/m \tag{28}$$

and for a sin Ct/Ct envelope pulse train at the half power point after the envelope peak if

$$\gamma = 4T/T_h = 4/m_h. \tag{29}$$

Substituting these conditions, (20)–(24) can be reduced to the form of (9), (10), (18), and (19). The numerical values of the constants c are given in Table I.

Appendix III

Storage Tube Filter

The noise power output in the storage tube representation of Fig. 9 will be determined by the gating sequence and by the feedback loop characteristics. Following the method given by Lawson and Uhlenbeck[20] it can be shown that the gating affects only the amplitude of the noise power spectrum without changing its spectral distribution. The noise power output of the storage tube filter P_{No} can be related to the noise power input P_{Ni} by multiplying a factor representing the noise suppression in the feedback loop with gating factors. This results in

$$P_{No} = \frac{k-1}{k^2} \frac{1-e^{-\gamma}}{1+e^{-\gamma}} P_{Ni} \tag{30}$$

assuming one read sweep per k sweeps of the storage tube.

The signal output of the storage tube will consist of a gated pulse train. The output waveform of the filter $V_{so}(t)$ is computed in time domain for an input signal of m rectangular pulses of pulse duration T_p.

$$V_{so}(t) = S(1-e^{-\gamma}) \left\{ \sum_{q=0}^{\langle m/k \rangle - 1} (a - be^{-qk\gamma}) + A \sum_{q=\langle m/k \rangle}^{\infty} e^{-qk\gamma} \right\} \cdot \{1[t-(k-1+qk)T] - 1[t - T_p - (k-1+qk)T]\} \tag{31}$$

where

$$a = [(1-e^{-\gamma})^{-1} - (1-e^{-k\gamma})^{-1}] 1(m-k) \tag{32}$$

$$b = e^{-k\gamma} a \tag{33}$$

$$A = e^{-k\gamma} \left[\frac{e^{m\gamma}-1}{1-e^{-\gamma}} - 1(m-k) \frac{e^{k\gamma \langle m/k \rangle} - 1}{1-e^{-k\gamma}} \right] \tag{34}$$

$1(x)$ is a unit step function and $\langle m/k \rangle$ designates the largest integer not exceeding (m/k). The peak signal output may occur during the output gates prior to or after the last pulse of the pulse train. Taking into account the low-pass characteristics,[21] the peak filter output during the gate prior to or during the last pulse of the pulse train is

$$V_{so} = 1.17S(1-e^{-\gamma})(a - be^{-\langle m/k \rangle k\gamma + k\gamma}) \tag{35}$$

with a and b given by (32) and (33). The peak filter output during the gate after the last pulse of the pulse train is

$$V_{so} = 1.17S(1-e^{-\gamma}) A e^{-\langle m/k \rangle k\gamma} \tag{36}$$

with A given by (34).

[19] The expression in the square brackets differs from the corresponding expression listed by Freedman and Margolin, *op. cit.*, by a constant factor, 1.17, which is dependent on the low-pass characteristics. Limiting the interval of integration for computing the peak signal output in (12) is equivalent to assuming a rectangular low-pass filter with a cutoff frequency $\omega_c = 2\pi/T_p$ in series with the comb filter. This low-pass filter gives rise to the factor 1.17 according to footnote 21.

[20] Lawson and Unlenbeck, *op. cit.*, pp. 253–254.

[21] E. A. Guillemin, "Communication Networks," John Wiley and Sons, Inc., New York, N. Y., vol. 2, pp. 485–486; 1949.

The signal power output is computed as in (4) after computing the signal spectrum $V_{so}^{i}(\omega)$ of the signal $V_{so}(t)$ in (31). This results in

$$P_{so} = P_{si}m^{-1}(1 - e^{-\gamma})^2\left[a^2\left\langle\frac{m}{k}\right\rangle - 2ab\,\frac{1 - e^{-(m/k)k\gamma}}{1 - e^{-k\gamma}} + b^2\,\frac{1 - e^{-2(m/k)k\gamma}}{1 - e^{-2k\gamma}} + A^2\,\frac{e^{-2k\gamma(m/k)}}{1 - e^{-2k\gamma}}\right] \quad (37)$$

where a, b, and A are given by (32), (33), and (34), respectively.

Appendix IV

Effects of Inaccurate Delay Times in the Simple Feedback Type Filter

A simple feedback type comb filter with a transfer function

$$H(j\omega) = (1 + Ae^{-j\omega T_k})^{-1} \quad (38)$$

and with inaccurate delay time T_k is considered in this appendix. The output of the filter consists of a weighted sum of past inputs given by

$$V_{so}(t) = V_{si}(t) + AV_{si}(t - T_k) + A^2V_{si}(t - 2T_k) + \cdots + A^{m-1}V_{si}[t - (m - 1)T_k]. \quad (39)$$

The delay time of the delay line is assumed to vary normally with the mean value equal to T and with the standard deviation equal to T_σ.

A pulse recirculating $(i - 1)$ times will have the distribution

$$(i - 1)(T_k - T) = N[0, (i - 1)T_\sigma] \quad (40)$$

for its deviation from the mean delay time, assuming a negligible change in the delay time, T_k, during the pulse storage. The input of the filter consists of rectangular pulses of amplitude S and of duration T_p. The time, t, during the last pulse of the pulse train measured relative to the leading edge of this pulse is defined as

$$\tau T_p = t - t_{le} \quad (41)$$

where

$$0 \leq \tau \leq 1. \quad (42)$$

The probability of the ith pulse to contribute to filter output at time t during the last pulse of the pulse train is

$$P_i = \frac{1}{\sqrt{2\pi}T_\sigma(i - 1)}\int_{-(1-\tau)T_p}^{T_p} e^{-1/2[x/(i-1)T_\sigma]^2}\,dx \quad (43)$$

where $i \geq 2$. Introducing the notation

$$F(t_0) = (2\pi)^{-0.5}\int_0^{t_0} e^{-0.5t^2}\,dt \quad (44)$$

(43) becomes

$$P_i = F\left[\frac{\tau T_p}{T_\sigma(i - 1)}\right] + F\left[\frac{(1 - \tau)T_p}{T_\sigma(i - 1)}\right]. \quad (45)$$

The average output of the filter at the time t defined by (41) is a weighted sum of the probabilities P_i plus one, multiplied by the amplitude of the input signal S

$$\overline{V_{so}(\tau T_p)} = S\left\{1 + \sum_{i=2}^{m} A^{i-1}F\left[\frac{\tau T_p}{(i - 1)T_\sigma}\right] + \sum_{i=2}^{m} A^{i-1}F\left[\frac{(1 - \tau)T_p}{(i - 1)T_\sigma}\right]\right\}. \quad (46)$$

For small values of m, the summations are evaluated directly. For larger values of m the summations are approximated by integrals.

Assuming inaccurate delay times of the delay line of the comb filter, signal power output will be decreased. The signal power output of the filter is

$$P_{so} = (2\pi)^{-1}(ST_p)^2\sum_{n=-Tf_p}^{(Tf_p)-1}\left(\frac{\sin 0.5\omega_nT_p}{0.5\omega_nT_p}\right)^2\cdot V \quad (47)$$

where V denotes a fourfold summation of integrals

$$V = \sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{q=0}^{m-1}\sum_{p=0}^{m-1} e^{-j\omega_n(i-k)T_e}A^{i+k} \cdot \int_{2\pi n/T}^{2\pi(n+1)/T} e^{-j\omega(i-k+q-p)T}\,d\omega, \quad (48)$$

ω_n is defined by

$$\omega_n = \frac{(2n + 1)\pi}{T} \quad (49)$$

and the probability distribution of the delay time error T_e is

$$P(T_e) = (2\pi)^{-0.5}T_\sigma^{-1}e^{-T_e^2/2T_\sigma^2}. \quad (50)$$

After computing the average value of V the average of the signal output power, P_{so}, is obtained from (47).

Appendix V

Crosscorrelators

In the crosscorrelation type filter, the input signal is multiplied with a locally generated noiseless reference signal, and the multiplier output is then integrated. The signal, $V_{si}(t)$, and the reference signal, $g(t)$, are both considered to be pulse trains of m pulses of duration T_p and of pulse-repetition period T. Thus

$$V_{si}(t) = S\sum_{i=0}^{m=1}[1(t - iT) - 1(t - iT - T_p)] = Sg(t). \quad (51)$$

Denoting the input noise by $V_N(t)$ an integrator following the multiplier will have an output

$$V(\tau) = \int_0^{mT}[V_{si}(t) + V_N(t)]g(t + \tau)\,dt. \quad (52)$$

The maximum signal output which occurs at $\tau = 0$ is

$$V_{\max} = V(0) = mST_p + \int_0^{mT} V_N(t)g(t)\,dt \quad (53)$$

assuming undistorted signal pulses. The idealized low-

ass filter of cutoff frequency ω_p spreads out the input ulses[21] and decreases their area during the undistorted eference pulses. This will decrease the signal output ndicated in (53) by a factor

$$C = (\pi T_p)^{-1} \int_0^{T_p} [Si(\omega_p t) - Si\omega_p(t - T_p)]\, dt$$

$$= \pi^{-2} \int_0^{2\pi} Si\, x\, dx \approx 0.9. \qquad (54)$$

The autocorrelation function of the noise after the multi-lication with the pulse train, $g(t)$, is the product of the espective autocorrelation functions

$$R_g(\tau) = \frac{2P_N(0)}{\tau} \frac{T_p - \tau}{T} \sin \omega_p \tau \qquad (55)$$

or $|\tau| \leq T_p$

$$R_g(\tau) \approx 0 \qquad (56)$$

or $|\tau| \geq T_p$. The mean square noise output of the filter corresponding to the instantaneous output of (53) is obtained as[18]

$$P_{N_o} = 2 \int_0^{mT} (mT - \tau) R_g(\tau)\, d\tau. \qquad (57)$$

Substituting $R_g(\tau)$ of (55) and (56) in (57), evaluating the integral, and assuming T_p much less than T

$$P_{N_o} = 4P_N(0)T_p m Si(2\pi). \qquad (58)$$

Using the signal output of (53) in conjunction with (54), the peak signal-to-rms-noise improvement figure of the crosscorrelator becomes

$$\frac{V_{so}}{E_{No}} \bigg/ \frac{V_{si}}{E_{Ni}} = 1.34\sqrt{m}. \qquad (59)$$

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Non-Mean-Square Error Criteria*

SEYMOUR SHERMAN†

***Summary*—While in the engineering literature non-mean-square error criteria for predictors are often presented as physically significant and then shunted aside because of mathematical unmanageability, it is shown here that in the case of Gaussian processes all such criteria given in three recent textbooks yield the same predictor as the linear minimum mean-square predictor of Wiener.**

IN the usual engineering presentation[1-3] of the Wiener theory of prediction, it has become conventional to make remarks relating to 1) Gaussian statistics, 2) linearity of the predictor, and 3) the mean-square error criterion. It is generally remarked[4] that while other error criteria are "physically appropriate," they are "apparently too difficult to handle mathematically" and the mean-square error "lends itself conveniently to mathematical analysis." With these regrets, the consequences of the mean-square error criterion become the limited subject of the investigation. Frequently it is remarked correctly[5] that in the presence of Gaussian statistics the mean square criterion yields an optimum predictor which is linear, so that no nonlinear predictor is better than the Wiener predictor in the case of Gaussian statistics and the mean-square error criterion. The purpose of this paper is to point out that the rejected error criteria are mostly of the form $E\phi(e)$, where e is a random variable, the error, while ϕ satisfies

$$0 \leqq \phi(e) = \phi(-e); \quad 0 \leqq e_1 \leqq e_2 \Rightarrow \phi(e_1) \leqq \phi(e_2) \qquad (1)$$

and in the presence of Gaussian statistics such an error criterion yields a linear predictor as the optimum predictor. This is the same linear predictor which is yielded by the mean-square error criterion. Thus, in the presence of Gaussian statistics, the results of the Wiener mean-square error theory of prediction, either operating on data from the infinite past or only the finite past, have a wider applicability than is usually realized; they apply automatically to almost all of the error criteria that have

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† Moore School of Elec. Eng., University of Pennsylvania, Philadelphia, Pa.
[1] J. G. Truxal, "Automatic Feedback Control System Synthesis," McGraw-Hill Book Co., Inc., New York, N. Y., pp. 410–499; 1955.
[2] J. H. Laning, Jr. and R. H. Battin, "Random Processes in Automatic Control," McGraw-Hill Book Co., Inc., New York, N. Y., pp. 172–182; 1956.
[3] W. B. Davenport, Jr. and W. L. Root, "An Introduction to the Theory of Random Signals and Noise," McGraw-Hill Book Co., Inc., New York, N. Y., pp. 219–221; 1958.
[4] Laning and Battin, *op. cit.*, p. 180.

[5] Davenport and Root, *op. cit.*, p. 231.

been proposed. The same integral equation yields the solution of all these cases.

The first lemma is well known.[6,7]

Lemma: If ϕ satisfies (1) and F is a probability distribution function on the reals which is symmetric and unimodal with mode at the origin so that $F(X) = 1 - F(-X)$ at each continuity point of F and F is convex for $x < 0$, then

$$\int \phi(X)\, dF(X) \leqq \int \phi(X - a)\, dF(X)$$

for each real a, when the integrals exist; if either integral diverges, the one on the right does.

It follows in particular that if the distribution function is absolutely continuous, then the lemma holds. We use only this version, and the reader can convince himself of its truth by drawing a diagram.

Now consider a Gaussian process $\{s_t: -\infty < t < \infty\}$ and the problem of finding the estimate $\hat{s}_f$, $f > 0$ in terms of $\{s_t: t \leqq 0\}$, which minimizes $E\phi(e)$, where

$$e = s_f - \hat{s}_f(\{s_t: t \leqq 0\}) \tag{2}$$

and ϕ satisfies (1). Note that

$$E\phi(e) = E[E(\phi(e) | \{s_t: t \leqq 0\})]. \tag{3}$$

Since the conditional distribution of s_f on the hypothesis $\{s_t: t \leqq 0\}$ is Gaussian and therefore unimodal and symmetric about the mode, $E(s_f | \{s_t: t \leqq 0\})$, we see that $\hat{s}_f = E(s_f | \{s_t: t \leqq 0\})$ minimizes $E(\phi(e) | \{s_t: t \leqq 0\})$ and therefore minimizes $E[E(\phi(e) | \{s_t: t \leqq 0\})] = E\phi(e)$. *Thus, in the case of a Gaussian process,*

$$\hat{s}_f = E(s_f \mid \{s_t: t \leqq 0\}),$$

which is the solution of the Wiener integral equation and is that predictor which minimizes $E(e^2)$, *also automatically minimizes* $E(\phi(e))$ *where* ϕ *satisfies* (1).

Among the ϕ satisfying (1) which have been offered in the literature and then rejected on account of mathematical complication are

$\phi_1(e) = | e |.$[8-10]

$$\phi_2(e) = \begin{cases} 0, & | e | < a, \\ 1, & | e | \geqq a. \end{cases}$$ [4,9]

$$\phi_3(e) = \begin{cases} 0, & | e | < a, \\ \dfrac{| e | - | a |}{| b | - | a |}, & | a | \leqq | e | \leqq | b |, \\ 1 \cdots 1, & | b | < | e |. \end{cases}$$ [11]

It should be further pointed out that considerations similar to those presented here apply to the case where prediction for several times $f_1 > f_2 > \cdots > f_n > 0$ is carried so that $E\phi(e_1, \cdots, e_n)$ is minimized. Particularly in the case of a Gaussian process, the two criteria (5.1–5) and (5.1–6) of Laning and Battin[4] yield the same optimum predictor.

In the case of a non-Gaussian process, where the conditional distribution of s_f given $\{s_t: t \leqq 0\}$ satisfies the conditions of the lemma, again the optimum predictor in the sense of minimizing $E\phi(e)$, where ϕ satisfies (1), is the same as the minimum mean-square error predictor, although there is no guarantee in this case that the predictor is linear.

It is a pleasure to acknowledge the stimulation of students in my course EE668, Engineering Statistics Seminar. On one occasion last semester I pointed out the well-known fact that the optimum predictor for least-square error criterion was conditional expectation[12] and that for Gaussian processes the conditional expectation was linear.[13] At this point, in response to a question by I. Maron, N. C. Randall indicated that for the absolute deviation as error criterion, as given by Cramer[14] the conditional median should be the optimum predictor. This exchange led the author to observe that condition (1) leads to a collection of manageable error criteria (for Gaussian processes) which included not only ϕ_1, but the previously rejected ϕ_2 and ϕ_3.

[6] T. W. Anderson, "The integral of a symmetric unimodal function over a symmetric convex set and some probability inequalities," *Proc. Amer. Math. Soc.*, vol. 6, pp. 170–176; 1955.

[7] S. Sherman, "A theorem of convex set with applications," *An. Amer. Math. Stat.*, vol. 26, pp. 763–767; 1955.

[8] Laning and Battin, *op. cit.*, pp. 179–180.

[9] Davenport and Root, *op. cit.*, p. 220.

[10] Truxal, *op. cit.*, p. 413.

[11] *Ibid.*, p. 414.

[12] H. Cramer, "Mathematical Methods of Statistics," Princeton University Press, Princeton, N. J., pp. 271–272; 1946.

[13] *Ibid.*, p. 315.

[14] *Ibid.*, p. 179.

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A Comment on Pattern Redundancy*

A recent paper by Glovazky[1] discussed a method for determination of redundancies in a given, finite set of P two-tone patterns such that each pattern could be described by a finite number C of cells, each all white or all black. Glovazky draws up a code schedule, consisting of the numbers 0 (white) and 1 (black) to describe the scanning process and effect separation.

Glovazky has shown that at most $P - 1$ cells are necessary to effect separation. The least possible number is the smallest integer y satisfying $y \geq \log_2 P$. One problem is the selection of the smallest number z of cells, $\log_2 P \leq z \leq P - 1$, that makes separation possible. Call these "special" cells.

Consider first the case $P = 2^y$. If we can find y special cells out of the C available cells, such that the reduced "code" schedule for these cells, arranged in descending value, forms all binary numbers from $P - 1$ to 0, then the minimum separation path has been found. An aid to finding these special cells is the following necessary condition:

The number of "ones" (also called "weight") in each cell column for these y cells is exactly P/2.

In the more general case, the following necessary condition for finding y separation cells can be used to aid in the selection of these special cells. Write out in ascending order the binary numbers from 0 to $(2^y - 1)$. Bracket the numbers from 0 to $P - 1$ inclusive, and count the number of "ones" in each bracketed column; denote these by $m_0, m_1, m_2 \cdots m_{y-1}$. Now bracket the numbers from $(2^y - 1)$ to $(2^y - P)$ inclusive, and count the number of "ones" in each bracketed column; denote these by $n_0, n_1, n_2 \cdots n_{y-1}$. Obviously, $m_j + n_j = P$. Then a necessary condition for finding y cells $C_0, C_1 \cdots C_{y-1}$ to effect separation is that cell column C_j contain c_j "ones," where $m_j \leq c_j \leq n_j$. This process is illustrated in Fig. 1.

Consider Glovazky's example, with $P = 6$, $C = 9$. In this case, we find $m_0 = 3$, $m_1 = 2$, $m_2 = 2$, hence $3 \leq c_0 \leq 3$ (see Fig. 1).

$$2 \leq c_1 \leq 4$$

$$2 \leq c_2 \leq 4.$$

In Glovazky's code schedule, cell columns 3, 5, and 9 each contain 3 "ones." Examination indicates that these three cells effect separation, and it is not necessary to use four cells.

Consider next Glovazky's alpha-numeric case with $C = 100$, $P = 35$. Here $y = 6$, $(P - 1) = 34$. Using Glovazky's method, we would find 34 or less cells to effect separation. To find 6 special cells—using trial and error—could prove a formidable task, since one can choose 6 cells out of 100 in about 10^9 ways. Using the necessary condition above, we find

$$17 \leq c_0 \leq 18$$

$$17 \leq c_1 \leq 18$$

$$16 \leq c_2 \leq 19$$

$$16 \leq c_3 \leq 19$$

$$16 \leq c_4 \leq 19$$

$$3 \leq c_5 \leq 32.$$

Hence trial should start with those cell columns containing 17 or 18 "ones." Conversely, if there is no column containing 17 or 18 "ones," then separation with six cells is not possible.

$$\begin{array}{ccc} 0&0&0\\ 0&0&1\\ 0&1&0\\ 0&1&1\\ 1&0&0\\ 1&0&1\\ 1&1&0\\ 1&1&1 \end{array}$$

n bracket — m bracket

(a) Bracketed binary number sequence.

$$\begin{array}{ccc} 0&0&0\\ 0&0&1\\ 0&1&0\\ 0&1&1\\ 1&0&0\\ 1&0&1\\ \hline 2&2&3\\ m_2&m_1&m_0 \end{array}$$

(b) Finding m_j.

$$\begin{array}{ccc} 0&1&0\\ 0&1&1\\ 1&0&0\\ 1&0&1\\ 1&1&0\\ 1&1&1\\ \hline 4&4&3\\ n_2&n_1&n_0 \end{array}$$

(c) Finding n_j.

Fig. 1.

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[1] A. Glovazky, "Determination of redundancies in a set of patterns," IRE Trans. on Information Theory, vol. IT-2, pp. 151–153; December, 1956.

Relative Efficiency of English and German Languages for Communication of Semantic Content*

Since the formulation of the statistical theory of communication[1,2] nearly a decade ago, interest in the statistical properties of language, the chief vehicle of communication, has been greatly stimulated for many practical reasons, such as the construction of efficient codes. The view that language itself may be regarded as a code for certain conceptual entities has been frequently emphasized in current linguistic discussions. This is largely due to the fact that Shannon's measure for the entropy of a stochastic process has been found to be readily applicable for the calculation of the selective information content of language considered as a set of discrete code symbols.

This view carries with it, however, certain implications which do not seem to have been explicitly stated or exploited. For, if we regard, by way of an illustration, the two statements, "All that glitters is not gold" in English and its translation, "Es ist nicht alles Gold was glanzt" in German as alternative codes having the same meaning or significance, it is immediately evident that a comparison between the two languages is possible with regard to the efficiency with which they encode the same semantic content[3] (into linguistic symbols) without making reference to its absolute measure. To use a metaphor translation from one language into another may be regarded as a transformation of the code which leaves the semantic content, but not the selective entropy of the code symbols, invariant. To be sure, the transformation is not unique—and this is significant—but comparison of different languages, considered as alternative codes, is still possible at a statistical level, and leads to interesting results. For instance, one may take a certain sample of text in English and also its translation, say, sentence by sentence, in German, and determine the number of bits of information represented by the original English text, N_E, and its translation in German, N_G, computed on the basis of, say, the letter frequencies obtained in the original and in the translation. N_E and N_G are the figures of interest if a decision is to be reached on the question whether the message should be sent in English or in German using optimal binary codes based on letter frequencies in these languages. More generally, by considering sufficiently long and semantically equivalent statements in different languages we can compare the relative communication efficiencies of the languages in several respects such as their efficiencies for transcription if their orthographies are different. A comparative study of some of the major languages of India on these lines is in progress here and is reported elsewhere.[4]

We shall confine ourselves here to a comparison of the English and German languages considered as alternate codes which carry the same semantic content. With this object, a number of articles were taken from

* Received by the PGIT, December, 1957.

[1] C. E. Shannon, "A mathematical theory of communication," *Bell Sys. Tech. J.*, vol. 27, pp. 379–423; July, 1948.

[2] N. Wiener, "Extrapolation, Interpolation and Smoothing of Stationary Time-series,'" John Wiley and Sons, Inc., New York, N. Y.; 1949.

[3] The phrase "semantic content" is used here as more or less synonymous with "meaning" as currently used in the literature on information theory.

[4] B. S. Ramakrishna, *et al.*, "Statistical studies in some Indian languages with applications to communication engineering," *J. Inst. Telecomm. Engrs.* (India), vol. 4, pp. 25–35; December, 1957.

recent numbers of *The Reader's Digest* for which sentence by sentence translations were readily available in its German edition. It is, of course, possible to obtain a rough estimate of the relative coding efficiencies of the two languages by merely counting the total number of letters n_E in the English text and n_G in the corresponding German text, and multiplying them, respectively, by the entropies H_E and H_G known on the basis of letter frequencies in these languages. In view of the rather short size of the sample (about 35,000 letters), it was decided to compute the letter frequencies actually obtained in the original and in the translation and calculate the entropies per letter appropriate to the texts under consideration.

TABLE I*

	English (original)	German (translation)	German (original)	English (translation)
Number of letters in sample, n	33,980	44,376	36,466	32,821
Number of word spaces in sample, s	7900	6579	6536	7381
Total of letters and spaces, $n + s$	41,880	50,955	43,002	40,202
Entropy disregarding word space, H	4.170	4.080	4.086	4.150
Entropy counting word space, H'	4.083	4.074	4.081	4.079
Total entropy without word space, $n \times H$	141,697	181,054	149,000	136,207
Total entropy with word space, $(n + s) H'$	171,017	207,590	175,491	163,984

*Number of letters, words, entropies per letter, *etc.*, of samples of English and German texts which are translations of each other and hence are semantically equivalent. The second and third columns give data relating to translation from English to German and the last two from German to English translation.

Articles from The Reader's Digest Which Were Used in the English to German Translation

	English (original)	*German (translation)*
[1]	"'Hot' Atoms—Industry's Versatile Detectives," (pp. 87–92, April, 1957).	"'Heisse' Atome," (pp. 69–72; April, 1957).
[2]	"What Makes a Woman Memorable," (pp. 35–36; April, 1957).	"Was eine Frau interessant macht," (pp. 35–36; May, 1957).
[3]	"Nightmare on the 79th Floor," (pp. 37–41; April, 1957).	"Inferno im 79. Stock," (pp. 57–62; May, 1957).
[4]	"Coping with the Compliment," (pp. 42–44; April, 1957).	"Wie man mit Komplimenten fertig wird," (pp. 132–140; May, 1957).
[5]	"They're Shipping Freight by Pipeline Now," (pp. 57–60; April, 1957).	"Pipelines haben Hochbetrieb," (pp. 68–72; May, 1957).
[6]	"Discovering the People of Paris," (pp. 61–62; April, 1957).	"Unter den Menschen von Paris," (pp. 73–75; May, 1957).
[7]	"Acne—Youth's Mysterious Enemy," (pp. 78–80; April, 1957).	"Akne—Kummer der Jugend," (pp. 48–51; May, 1957).
[8]	"Work!" (pp. 85–86; April, 1957).	"Arbeite," (pp. 84–86; May, 1957).

Articles Used in the German to English Translation

	German (original)	*English (translation)*
[1]	*Das Leiden eines Knaben*, (Bilingual Series), Conrad Ferdinand Meyer, George G. Harrap & Co., Ltd., London, Eng., pp. 4–10, 54–64; 1949.	*The Tribulations of a Boy*, translated by E. M. Huggard.
[2]	*Deutscher Stahlbau sein Feld ist die Welt*, Deutscher Stahlbau-Verband, Koln, Ger., "Die Welt, die Stahlbauingenieure sehn," pp. 12–19.	*German Steel Construction, Its Field Is the World*, "The world that civil engineers see."
[3]	*Chemische Grundbegriffe*, Alfred Benrath, George G. Harrap & Co., Ltd., London, Eng., "Wesen und Wege der Naturforschung," pp. 4–10.	*The Fundamental Ideas of Chemistry*, translated by Jethro Bithell, "The nature and the methods of scientific research."

In the second and third columns of Table I are given in order the number of letters n, the number of word spaces s, their sum $n + s$, the entropy H disregarding the word space, the entropy H' counting the word space as a symbol, and the total number of bits of information $n \times H$ and $(n + s)\ H'$ in these two cases for the English (original) and the German (translation) texts. Naive comparison at this stage shows that if we ignore the word space (next to last row), in German we require 181,054 bits as against the 141,697 bits required in English to express the same semantic content. It appears that about 30 per cent more bits of information are required in German than in English, or to put it more concisely, the semantic value of a "German-bit" is about 0.78 of an "English-bit." The comparison, however is unfair to German in two respects. First, it is well known that the word space has a significance in communication and is not altogether irrelevant. If we, therefore, take the word space into account we notice from the last row that German fares better than before, a German-bit being equal to 0.82 of an English-bit. Second, there is the possibility that the fact that the original text was in English could have given an advantage to English. This may be argued on the basis that when one does translation, there is a certain freedom of choice among alternative expressions possible in the language into which translation is made and that this uncertainty which is equivalent to noise[1] (in the sense of the information theory) must be overcome by the use of additional bits of information.

To check the validity of this argument we considered samples of texts with the original in German and the translation in English, the material being drawn from various sources. The corresponding figures for the number of letters, the entropies, etc., for German and English are given in the last two columns of the same table. We notice from the last two rows of these columns that a different ratio obtains between a German and an English-bit, and that, in particular, taking the word space as a symbol (last row) one German-bit is equal to about 0.94 of an English-bit. A bit in German even now carries somewhat less semantic content than a bit in English, but the advantage now rests with German.

In this situation we naturally ask whether it is possible to find a more exact relation between the semantic value of an English-bit and a German-bit. If we can assume that the extent of uncertainty in translation from German to English is the same as that from English to German then this relationship can be readily established. (Such an assumption appears reasonable in the case of two closely related languages like English and German though it may not hold good for widely different languages.) We merely have to make allowance for the additional number of bits required to overcome the "noise" in the total number of bits required in the language of translation. Making this assumption, let one bit of English be equal to x bits of German and let y bits be the noise accompanying each bit of information. Considering the word space as equivalent to a letter symbol, we can form from the last row the two equations

$$171{,}017\ (x + y) = 207{,}590$$

$$175{,}491\ (x^{-1} + y) = 163{,}984$$

for the English to German and the German to English translations. Solving for x and y we obtain

$$x = 1.149 \text{ and } y = 0.065.$$

From the small samples studied, it thus appears that the English language is somewhat more efficient than the German language in encoding semantic content into linguistic symbols to the extent that one bit of information in English carries as much semantic content as 1.15 bits of information in German. It appears also that the process of translation is generally accompanied by some degree of uncertainty and to overcome this one requires additional

bits of information. The authors wish to point out the desirability of using larger samples of more varied character to obtain more precise figures. The tedious nature of the task of counting the different letters (in the absence of mechanical aids), however, limited the authors to the present size of the sample.

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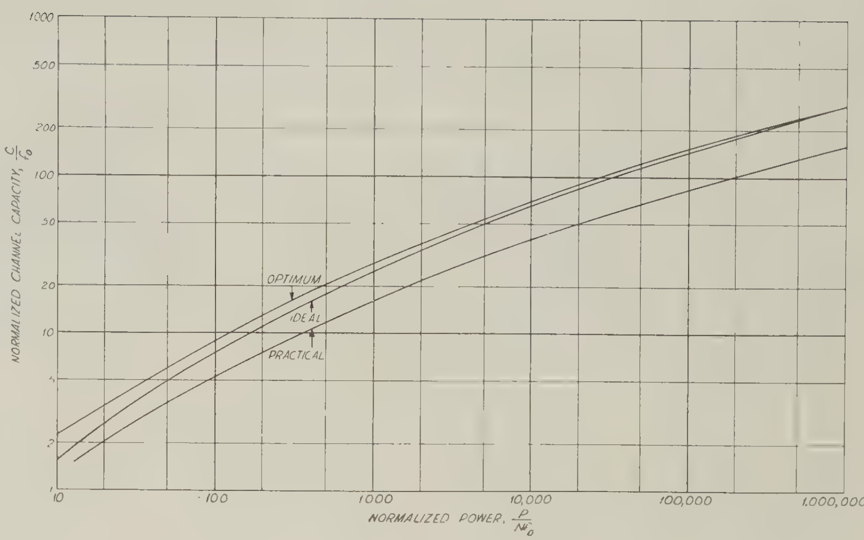

Fig. 1.

The Optimal Distribution of Signal Power in a Transmission Link Whose Attenuation Is a Function of Frequency*

When a transmission line having attenuation and noise is used as an element of a communication channel, the channel capacity depends on the nature of the signal. If the transmission line is linear and the noise is Gaussian noise, a formula for the channel capacity can be written explicitly. If the total signal power is limited, there is an upper bound to the channel capacity, and the maximum channel capacity can be approached only with signals having a certain frequency spectrum.

In practical cases, the frequency spectrum of the signal may be dictated by practical requirements other than information theory. For example, a flat band-limited spectrum may be desirable.

Furthermore, the effective channel capacity may be limited not by the theoretical upper bound, but by the signal-to-noise ratio in the worst part of the frequency spectrum. This is true, for example, in frequency multiplex systems, where a multiplex group is good if every channel is quiet enough to use, but hardly any better if some channels are much quieter than the minimum standard.

It is interesting to see in a simple example that the effective channel capacity is not much less with these restrictions on the signal spectrum than when the signal spectrum has its optimum shape. If the noise spectrum is flat with frequency, the attenuation is proportional to the square root of frequency, and the signal-to-noise ratio large, then the channel capacity using a flat signal spectrum has asymptotically the same upper bound as the channel capacity with the optimal signal spectrum. If the effective channel capacity is calculated on the assumption that the effective signal-to-noise ratio is the signal-to-noise ratio in the noisiest part of the band, then the upper bound of the effective channel capacity is lower by a factor 4/9 than in the optimal case.

* Received by the PGIT, October 1, 1957.

Let us suppose that the transmission link in question has an input and an output. Let us suppose that the attenuation suffered by a signal of frequency f in passing from the input to the output is $\alpha(f)$. Let us suppose that the noise emerging from the output is Gaussian noise with a power spectrum $N(f)$. Suppose finally that a signal is inserted into the input which has a power spectrum $P(f)$.

At the output, the signal power spectrum is

$$P'(f) = P(f)e^{-2\alpha(f)}. \qquad (1)$$

The upper limit of the channel capacity[1] is

$$C = \int_0^\infty \log\left(1 + \frac{P(f)e^{-2\alpha(f)}}{N(f)}\right) df. \quad (2)$$

The total signal power is

$$P = \int_0^\infty P(f)\, df. \qquad (3)$$

Following Shannon,[2] it is clear that the maximum of channel capacity for constant P occurs when

$$\begin{aligned} P(f) &= k - N(f)e^{2\alpha(f)} \cdots \quad \text{if this expression is greater than zero,} \quad &(4)\\ &= 0 \quad \text{otherwise,} \end{aligned}$$

where k is a constant chosen to give the desired value of total signal power P.

Now, to be specific, let

$$N(f) = N = \text{constant}; \qquad (5)$$

$$\alpha(f) = \sqrt{f/4f_0}. \qquad (6)$$

[1] C. E. Shannon, "Communication in the presence of noise," Proc. IRE, vol. 37, pp. 10–21; January, 1949.
[2] *Ibid.*, or see E. Goursat, "Cours d'Analyse," Paris, France, 2nd ed., vol. 3, pp. 575–576; 1915.

This attenuation law is representative of the high-frequency behavior of any transmission line carrying a TEM wave in which resistance is controlled by skin effect, *e.g.*, twisted pairs, coaxial lines, and so forth. The quantity f_o is simply a factor to normalize the unit of frequency. It is the frequency at which transmission through the link is one neper more attenuated than at zero frequency, *i.e.*, the "one-neper fall-off frequency."

In this case the signal power spectrum is

$$\begin{aligned} P(f) &= k - Ne^{\sqrt{f/f_o}} \quad f \le B, \quad &(7)\\ &= 0 \quad f \ge B; \end{aligned}$$

where B is described in terms of a new parameter u as follows:

$$B = [\log^2(u+1)]/4\alpha^2(1), \qquad (8)$$

$$u = (k - N)/N. \qquad (9)$$

The parameter u is defined as the signal-to-noise ratio at zero frequency. The integrals for total signal power P and the upper limit of channel capacity C can be evaluated to give

$$P/Nf_0 = (u+1)\log^2(u+1) - 2(u+1)\log(u+1) + 2u \quad (10)$$

$$C/f_0 = [\log^3(u+1)]/3. \qquad (11)$$

Although it is impractical to solve directly for C as a function of P, it is quite feasible to represent P/Nf_o and C/f_o parametrically in terms of u. The results are plotted in Fig. 1 to show how C varies with P.

Now consider another transmission system where the signal power is distributed uniformly over a transmission band of finite width. As before, the signal-to-noise ratio at the receiving end of the link is a function of frequency, and we must distinguish two cases: the "ideal" case, where the coding takes advantage of the frequency depend-

ence of S/N; and the "practical" case where the S/N at the top of the transmission band (or at the noisiest frequency) is accepted as the working S/N for the whole transmission band.

In both of these cases

$$P(f) = P/B \tag{12}$$

where B is the bandwidth of the channel. It will be seen later that the optimum bandwidth B is different in the two cases.

In the "ideal" case

$$C = \int_0^B \log\,[1 + (P/BN)e^{-\sqrt{f/f_0}}]\,df. \tag{13}$$

If we assume that the signal-to-noise ratio is large, the integral can be estimated by ignoring the unity in the argument of the logarithm. If we introduce again the zero frequency signal-to-noise ratio

$$u = P/BN \tag{14}$$

we find that C has a maximum when

$$B = f_0(\log u - 1)^2. \tag{15}$$

For this value of u, the values of normalized channel capacity and normalized power are

$$C/f_0 \simeq \tfrac{1}{3}[\log u - 1]^2[\log u + 2], \tag{16}$$

$$P/Nf_0 = u(\log u - 1)^2. \tag{17}$$

At the upper edge of the band

$$S/N = (P/BN)e^{-\sqrt{B/f_0}} = e, \tag{18}$$

i.e., no matter what the signal power, the bandwidth is such that the S/N is one neper at the highest frequency in the pass band. Of course, this estimate is not exact.

In the "practical" case, we assume that the effective S/N for the whole band is the S/N at the highest frequency, *i.e.*,

$$S/N = (P/BN)e^{-\sqrt{B/f_0}}. \tag{19}$$

In this case the *effective* upper limit of channel capacity is

$$C = \int_0^B \log\,[1 + (P/BN)e^{-\sqrt{B/f_0}}]\,df. \tag{20}$$

To the same degree of approximation as before

$$B = (4/9)(\log u - 1)^2 \tag{21}$$

$$C/f_0 \simeq (4/27)(\log u - 1)^2 \cdot(\log u + 1) \tag{22}$$

$$P/Nf_0 = (4/9)u(\log u - 1)^2. \tag{23}$$

Also, at the upper edge of the band, the signal-to-noise ratio is

$$S/N = u^{1/3}e^{2/3}. \tag{24}$$

Fig. 1 shows the normalized upper bound of channel capacity C/f_0 plotted as a function of normalized signal power P/Nf_0. The "optimum" system (shaped signal spectrum) is better than the other two, but not much: for large signal power, the "ideal" system (flat signal spectrum) is nearly as good, and the "practical" system (effective S/N equals S/N at the highest frequency transmitted) is worse only by a factor 4/9.

These examples suggest that the upper bound of the effective channel capacity of a transmission link may not be much affected by certain constraints on the signal spectrum. Hence, we should not hope to make big improvements in transmission system efficiency by altering signal spectra to increase channel capacity. Of course, each case must rest on its own merits.

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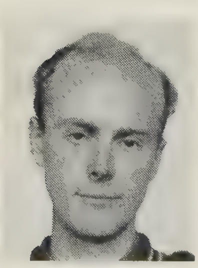

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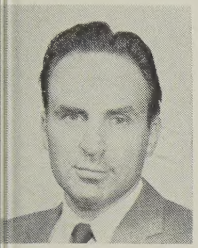

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While attending I.I.T. he worked for the Cook Research Laboratory on fire control problems, radar, and communication systems.

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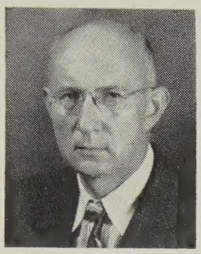

M. J. E. Golay

From 1924 until 1928 Dr. Golay was at the Bell Telephone Laboratories. In 1928 he went to the University of Chicago, where he obtained the Ph.D. degree in physics in 1931.

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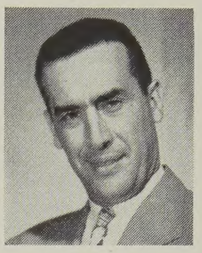

S. Sherman

From 1940–1941 he was a member of the Institute for Advanced Study, Princeton, N. J., and from 1941–1942 he taught mathematics and mechanics at the U. S. Naval Academy. He was at the Research Laboratory, Curtiss Wright Corporation, Buffalo, N. Y., from 1942 to 1944, and then spent one year with the Allegeny Ballistics Laboratory, Cumberland, Md., receiving the Naval Ordnance Development Award. From 1945 to 1948 he was an assistant professor in mathematics at the University of Chicago. In 1948 he returned to the Institute for Advanced Study, and from 1950 to 1954 he was with the Lockheed Aircraft Corporation, Burbank, Calif., in the Military Operations Research Division.

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